# SHARP LORENTZ-NORM ESTIMATES FOR DIFFERENTIALLY SUBORDINATE MARTINGALES AND APPLICATIONS

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Abstract. Let $1 < p < q \leq 2$. The paper contains the identification of the best constant $C_{p,q}$ such that the following holds. If $X$, $Y$ are Hilbert-space valued martingales such that $Y$ is differentially subordinate to $X$, then we have

$$\|Y\|_{p,\infty} \leq C_{p,q}\|X\|_{p,q}.$$

The proof rests on the careful combination of Burkholder's method and optimization arguments. As an application, related sharp Lorentz-norm inequalities for a wide class of Fourier multipliers are obtained.

## 1. Introduction

The motivation for the results obtained in this paper comes from a very natural question arising in martingale theory. To present this question from an appropriate perspective, let us start with the necessary background and notation. In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a complete probability space, equipped with a continuous-time right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the events $A$ with $\mathbb{P}(A) = 0$. Assume further that $X$, $Y$ are two continuous-time cadlag martingales, adapted to $(\mathcal{F}_t)_{t\geq 0}$, taking values in a separable Hilbert space $\mathcal{H}$. The norm and scalar product in $\mathcal{H}$ will be denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively; with no loss of generality, we may and do assume that $\mathcal{H} = \ell_2$ or $\mathcal{H} = \ell_2^n$ for some integer $n$. The symbol $[X,X]$ will stand for the quadratic variation (square bracket) of $X$, given by $[X,X] = \sum_{n\geq 1}[X^n, X^n]$, where $X^n$ is the $n$-th coordinate of $X$ and $[X^n, X^n]$ is the usual square bracket of the real-valued martingale $X^n$ (see Chapters VI and VII in Dellacherie and Meyer [13] or Chapter 4 in Métivier [20] for details).

We will impose a certain domination principle on the processes $X$, $Y$ under investigation. Following Bañuelos and Wang [6] and Wang [31], we say that the martingale $Y$ is differentially subordinate to $X$, if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is non-decreasing and nonnegative as a function of $t$. Let us discuss a few examples. If we treat two discrete-time martingales $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ as continuous-time processes (via $X_t = f_{\lfloor t\rfloor}$ and $Y_t = g_{\lfloor t\rfloor}$, $t \geq 0$), then the above domination amounts to saying that, almost surely,

$$|dg_n| \leq |df_n|, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

which is the original definition of the differential subordination, introduced by Burkholder in the eighties (cf. [11]). Here $df = (df_n)_{n\geq 0}$, $dg = (dg_n)_{n\geq 0}$ are the difference sequences of $f$ and $g$, given by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$, $n \geq 1$, with a similar definition of $dg$.

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2010 *Mathematics Subject Classification.* Primary: 60G44, 60G42; Secondary: 46E30.

*Key words and phrases.* BMO function, Lorentz norm, best constant, Burkholder's method, Fourier multiplier.

There is an important class of discrete-time processes for which (1.1) holds. We say that a martingale $g$ is the transform of a martingale $f$ by a predictable sequence $v$, if for any $n \geq 0$ we have the identity $dg_n = v_n df_n$. Here by predictability we mean that for each $n$, the random variable $v_n$ is measurable with respect to $\mathcal{F}_{(n-1)\vee 0}$. Note that if each term $v_n$ takes values in the interval $[-1, 1]$, then the condition (1.1) is satisfied.

We turn our attention to examples in continuous time. Suppose that $X$ is an arbitrary martingale and $H$ is a predictable process taking values in the interval $[-1, 1]$. Then the martingale $Y = H \cdot X$, given by the stochastic integral

$$Y_t = H_0 X_0 + \int_{0+}^t H_s \mathrm{d}X_s, \qquad t \geq 0, \tag{1.2}$$

is differentially subordinate to $X$. This follows immediately from the identity

$$[X, X]_t - [Y, Y]_t = (1 - |H_0|^2) X_0^2 + \int_{0+}^t (1 - |H_s|^2) \mathrm{d}[X, X]_s, \qquad t \geq 0.$$

Obviously, this example is just the continuous-time extension of the context of martingale transforms discussed above. Indeed, for any $f$, $g$, $v$ as previously, one considers the embedding $X_t = f_{\lfloor t \rfloor}$, $Y_t = g_{\lfloor t \rfloor}$, $H_t = v_{\lfloor t \rfloor}$, $t \geq 0$, and verifies the identity $Y = H \cdot X$.

As the final example, consider a Brownian motion $B$ in $\mathbb{R}^d$ and let $H$, $K$ be two predictable processes with values in $d \times d$ matrices, satisfying $\|H_t\|_{HS} \geq \|K_t\|_{HS}$ for all $t > 0$ (here $\|\cdot\|_{HS}$ stands for the Hilbert–Schmidt norm). Then the $\mathbb{R}^d$-valued martingales

$$X_t = \int_{0+}^t H_s \cdot \mathrm{d}B_s \quad \text{and} \quad Y_t = \int_{0+}^t K_s \cdot \mathrm{d}B_s, \qquad t \geq 0,$$

satisfy the differential subordination, since

$$[X, X]_t - [Y, Y]_t = \int_0^t \left( \|H_s\|_{HS}^2 - \|K_s\|_{HS}^2 \right) \mathrm{d}s, \qquad t \geq 0.$$

The differential subordination implies many interesting estimates between $X$ and $Y$, which can be further applied in various problems of harmonic analysis, e.g., in the study of sharp inequalities for Fourier multipliers in the Euclidean and non-Euclidean settings. The literature on the subject is very extensive, so we will only discuss below several selected results closely related to the theme of this paper. For the more detailed and systematic exposition on martingale inequalities, we refer the reader to the monograph [22]; for the applications, see e.g. [3, 6, 7, 14, 17, 25] and consult the references therein.

Probably the most prominent result in the area is the following sharp $L^p$ estimate, established by Burkholder in his seminal paper [11] in the discrete-time context. The continuous-time extension presented below is taken for Wang's paper [31].

**Theorem 1.1.** *Suppose that $X$, $Y$ are $\mathcal{H}$-valued martingales such that $Y$ is differentially subordinate to $X$. Then for any $1 < p < \infty$ we have the inequality*

$$\|Y_t\|_p \leq (p^* - 1) \|X_t\|_p, \qquad t \geq 0, \tag{1.3}$$

*where $p^* = \max\{p, p/(p-1)\}$. The constant $p^* - 1$ is the best possible, even for $\mathcal{H} = \mathbb{R}$: for any $1 < p < \infty$ and any $C < p^* - 1$, there exist $t \geq 0$ and a pair $(X, Y)$ of differentially subordinate real-valued martingales such that $\|Y_t\|_p > C \|X_t\|_p$.*

This result can be extended in many directions. For example, one can ask about the best constants in the corresponding weak-type estimates. As shown by Burkholder [11] and Suh [29], if $1 \leq p < \infty$, then we have

$$\mathbb{P}(|Y_t| \geq 1)^{1/p} \leq c_p \|X_t\|_p, \qquad t \geq 0,$$

where the optimal constant is given by $(2/\Gamma(p+1))^{1/p}$ if $1 \leq p \leq 2$ and $\left(p^{p-1}/2\right)^{1/p}$ for remaining $p$. There is an alternative version of this result, established in [24], which refers to a different norming of weak $L^p$ spaces. For any $1 < p < \infty$ and an arbitrary random variable $\xi$ with values in $\mathcal{H}$, define

$$\|\xi\|_{p,\infty} = \sup\left\{\mathbb{P}(A)^{1/p-1}\int_A |\xi| \mathrm{d}\mathbb{P}\right\}, \tag{1.4}$$

where the supremum is taken over all events $A$ of positive probability. Then, if $Y$ is differentially subordinate to $X$ and $1 < p < \infty$, we have the sharp estimate

$$\|Y_t\|_{p,\infty} \leq k_p \|X_t\|_p, \qquad t \geq 0, \tag{1.5}$$

where $k_p = (\Gamma((2p-1)/(p-1)))^{1/p}$ if $1 < p \leq 2$ and $k_p = c_p = \left(p^{p-1}/2\right)^{1/p}$ for $p \geq 2$. One can investigate other families of estimates, including logarithmic, exponential and restricted weak-type bounds (cf. [23, 25]); one can also consider maximal versions of such results, as well as various modifications involving certain additional boundedness conditions on $X$ and $Y$: see [22] for an overview. There is a powerful technique, invented by Burkholder [11], which is very efficient in the study of such problems. Roughly speaking the approach rests on the construction of an appropriate special function, enjoying certain size and concavity requirements; we will discuss the technique briefly in Section 2. However, the method has its limitations and allows the study of only those estimates, which can be expressed in an appropriate integral form (see (2.1) below). On the other hand, there are many function spaces in which the corresponding norms have a more involved structure. In our considerations below, we will be concerned with a class of Lorentz-norm estimates. For a random variable $\xi$, let $\xi^* : (0,1] \to [0,\infty)$ stand for its decreasing rearrangement, defined by

$$\xi^*(t) = \inf\Big\{\lambda \geq 0 : \ \mathbb{P}(|\xi| > \lambda) \leq t\Big\}.$$

Then for any $0 < p, q < \infty$, we define the Lorentz space $L^{p,q} = L^{p,q}(\Omega, \mathcal{F}, \mathbb{P})$ as the family of all (equivalence classes of) random variables $\xi$ for which

$$\|\xi\|_{p,q} = \left(\int_0^1 (t^{1/p}\xi^*(t))^q \frac{\mathrm{d}t}{t}\right)^{1/q}.$$

Our main result can be stated as follows. In our considerations below, we use the weak norming (1.4), and the symbol $p'$ is the Hölder conjugate to $p$, i.e., $p' = p/(p-1)$.

**Theorem 1.2.** *Suppose that $1 < p \leq q \leq 2$. Let $X$, $Y$ be two $\mathcal{H}$-valued martingales such that $Y$ is differentially subordinate to $X$. Then we have the estimate*

$$\|Y_t\|_{p,\infty} \leq 2^{-1/p'}(p'/q')^{1+1/q'}\Gamma(q'+1)^{1/q'}\|X_t\|_{p,q}, \qquad t \geq 0. \tag{1.6}$$

*The constant is the best possible, even in the context of stochastic integrals* (1.2) *and for $\mathcal{H} = \mathbb{R}$.*

It should be emphasized that Burkholder's method is not applicable here directly: in the next section we describe the main idea behind our approach, in particular, we present the obstacles arising due to the appearance of Lorentz norms and outline the argument which will allow us to overcome these difficulties. Section 3 contains the proof of an auxiliary estimate which is dual to (1.6), while Section 4 is devoted to the proof of Theorem 1.2. The final section contains applications to wide class of Fourier multipliers, including the real part of Beurling–Ahlfors operator and more general class of linear combinations of second-order Riesz transforms.

## 2. On the approach

2.1. **Burkholder's method.** As we mentioned in the previous section, inequalities for differentially subordinated martingales can be handled by constructing certain special functions. The idea behind this method is the following. Suppose that $D$ is an open domain contained in $\mathcal{H}\times\mathcal{H}$, such that $(0,0)\in D$ (typically, $D$ is $\mathcal{H}\times\mathcal{H}$ itself or $D=\mathcal{B}\times\mathcal{H}$, where $\mathcal{B}$ is the unit ball of $\mathcal{H}$). Assume further that $V:D\to\mathbb{R}$ is a given Borel function and we are interested in the estimate

$$\mathbb{E}V(X_t,Y_t)\leq 0,\qquad t\geq 0, \tag{2.1}$$

for all pairs $(X,Y)$ of martingales taking values in $D$, such that $Y$ is differentially subordinate to $X$. To study such a problem, one searches for a special function $U$ on $D$, which satisfies the following three requirements:

1° We have $U(x,y)\leq 0$ for all $(x,y)\in D$ such that $|y|\leq|x|$.
2° We have $U\geq V$.
3° For any $t\geq 0$ and any pair $(X,Y)$ as above, we have $\mathbb{E}U(X_t,Y_t)\leq\mathbb{E}U(X_0,Y_0)$.

The existence of such a function $U$ immediately yields (2.1): we have $\mathbb{E}V(X_t,Y_t)\leq\mathbb{E}U(X_t,Y_t)\leq\mathbb{E}U(X_0,Y_0)\leq 0$, where the first passage follows from the majorization condition 2°, the second is due to 3° and the final follows from 1° (and the estimate $|Y_0|\leq|X_0|$, by the differential subordination). While the conditions 1° and 2° are simple pointwise estimates which can be investigated directly, the third requirement seems more intricate to analyze. It is more or less clear that this condition is related to some sort of concavity of $U$. For instance, 3° holds if $U$ is concave on $D$, but, roughly speaking, it is enough to assume the concavity in certain directions. Precisely, we have the following fact, which is a slight modification of the results of Wang [31] (see Proposition 2 there). In what follows, for any $r>0$, the symbol $rD$ stands for the dilated set $\{(rx,ry):(x,y)\in D\}$.

**Lemma 2.1.** *Let $U:D\to\mathbb{R}$ be a continuous function which is of class $C^1$ in the interior of D and of class $C^2$ on $D_i$, where $D_1$, $D_2$, ..., $D_m$ are open subsets of $D$ such that $\overline{D_1}\cup\overline{D_2}\cup\ldots\cup\overline{D_m}=D$. Assume in addition that there is a Borel function $c:D_1\cup D_2\cup\ldots\cup D_m\to[0,\infty)$ satisfying*

$$\sup_{(x,y)\in(D_1\cup D_2\cup\ldots\cup D_m)\cap rD} c(x,y)<\infty\qquad \textit{for all } 0<r<1 \tag{2.2}$$

*and such that for all $(x,y)\in D_1\cup D_2\cup\ldots\cup D_m$ with $|x||y|\neq 0$ and all $h\in\mathbb{R}$, $k\in\mathcal{H}$,*

$$\langle U_{xx}(x,y)h,h\rangle+2\langle U_{xy}(x,y)h,k\rangle+\langle U_{yy}(x,y)k,k\rangle\leq -c(x,y)(|h|^2-|k|^2). \tag{2.3}$$

*Let $(X,Y)$ be a martingale pair taking values in $rD$ for some $0<r<1$, such that $Y$ is differentially subordinate to $X$. Then for any $t\geq 0$ there is a non-decreasing sequence*

$(\tau_n)_{n\geq 0}$ *of stopping times converging to* $\infty$ *almost surely, such that*

$$\mathbb{E}U(X_{\tau_n\wedge t},Y_{\tau_n\wedge t})\leq \mathbb{E}U(X_0,Y_0). \tag{2.4}$$

Though the formulation of the above lemma might look a little complicated, its meaning is very simple: up to some boundedness requirements (expressed in (2.2)) and straightforward limiting arguments (which involve letting $r\to 1$ in the requirement $(X,Y)\in rD$ and $n\to\infty$ in the assertion (2.4)), the desired condition 3° is a consequence of the concavity inequality (2.3). Putting all the above observation together, we see that the validity of the estimate (2.1) can be deduced from the existence of a function $U$ on $D$, which satisfies the pointwise inequalities 1°, 2° and (2.3).

The method described above has been successful in the study of various sharp estimates for differentially subordinate martingales. For example, the choice $V(x,y)=|y|^p-C_p^p|x|^p$ corresponds to the moment inequality (1.3); Burkholder proved that the corresponding special function $U$ is given by

$$U(x,y)=\alpha_p(|y|-(p^*-1)|x|)(|x|+|y|)^{p-1},$$

where $\alpha_p$ is a certain positive constant depending only on $p$. As another example, functions of the form $V(x,y)=(|y|-\alpha)_+-\beta|x|^p$ for some $\alpha$, $\beta>0$ lead to weak-type estimates. Let us discuss this connection in a more detailed manner, as we will need similar arguments later on. For simplicity, let us restrict ourselves to $p\geq 2$ (analogous, but slightly different and more complex calculations work for $1<p<2$ as well). As shown in Section 2.6 of [5], we have the inequality

$$\mathbb{E}\left(|Y_t|-1+\frac{1}{p}\right)_+\leq \frac{p^{p-2}}{2}\mathbb{E}|X_t|^p,\qquad t\geq 0,$$

which corresponds to the above function $V$ with $\alpha=1-1/p$ and $\beta=p^{p-2}/2$. The special function $U$ has quite a complicated formula and we will not present it here: we refer to [5] for details. Fix an auxiliary parameter $\lambda>0$ and apply the above estimate to the martingale pair $(X(1-p^{-1})/\lambda,Y(1-p^{-1})/\lambda)$: note that the differential subordination of the processes is inherited from $(X,Y)$. After some straightforward manipulations, we obtain

$$\mathbb{E}(|Y_t|-\lambda)_+\leq \frac{(p-1)^{p-1}\lambda^{1-p}}{2p}\mathbb{E}|X_t|^p,\qquad t\geq 0.$$

Pick an arbitrary event $A$ of positive probability. By the estimate above, we have

$$\begin{aligned}\int_A|Y_t|\mathrm{d}\mathbb{P}&=\int_A(|Y_t|-\lambda)\mathrm{d}\mathbb{P}+\lambda\mathbb{P}(A)\\&\leq \mathbb{E}(|Y_t|-\lambda)_++\lambda\mathbb{P}(A)\leq \frac{(p-1)^{p-1}\lambda^{1-p}}{2p}\mathbb{E}|X_t|^p+\lambda\mathbb{P}(A).\end{aligned}$$

Minimizing the right-hand side with respect to $\lambda$ gives

$$\int_A|Y_t|\mathrm{d}\mathbb{P}\leq \frac{p^{p-1}}{2}\|X_t\|_p\,\mathbb{P}(A)^{1-1/p},$$

which is the weak-type bound.

We see that the above method enables the study of only those estimates, which can be rewritten in the 'integral' form (2.1). Since the norms in the Lorentz spaces $L^{p,q}$ are not of that shape (unless $p=q$ or $q=\infty$), some new ideas need to be developed. Let us sketch the main steps of our approach.

2.2. **A dual estimate.** It turns out that it is convenient to study first a related dual result (the reason for this will be clarified in a moment). For the sake of brevity, let

$$C_{p,q} = 2^{-1/p'} \, (p'/q')^{(q'+1)/q'} \, \Gamma(q'+1)^{1/q'}$$

be the constant appearing in (1.6). The associated dual estimate reads

$$\|Y_t\|_{p,q} \leq C_{p',q'} \|X_t\|_1^{1/p} \|X_t\|_\infty^{1/p'}, \tag{2.5}$$

for $2 \leq q < p < \infty$ and $t \geq 0$. Here, as before, $X$, $Y$ are $\mathcal{H}$-valued martingales such that $Y$ is differentially subordinate to $X$. Fix $t$ and note that in (2.5), we may assume $\|X_t\|_\infty = 1$, by homogeneity. Now the argument splits into two steps. The first part is the following. Suppose that $\Phi$, $\Psi$ are some Young functions such that for all $u \geq 0$ and $v > 0$ we have

$$u^q v^{q/p-1} \leq \Phi(u) + \Psi(v). \tag{2.6}$$

Then the direct integration yields

$$\|Y_t\|_{p,q}^q = \int_0^1 (Y_t^*(s))^q s^{q/p-1} \mathrm{d}s \leq \int_0^1 \Phi\big(Y_t^*(s)\big)\mathrm{d}s + \int_0^1 \Psi(s)\mathrm{d}s. \tag{2.7}$$

Here the assumption $q < p$ is crucial: then $(Y^*(s))^q$ and $s^{q/p-1}$ are equimonotone and Young's inequality can be applied. Note that the latter integral $\int_0^1 \Psi(s)\mathrm{d}s$ is deterministic. The second step is to prove the sharp bound of the form

$$\int_0^1 \Phi\big(Y_t^*(s)\big)\mathrm{d}s \leq c_{p,q}\|X_t\|_1, \tag{2.8}$$

for some $c_{p,q} > 0$. Observe that this estimate can be rewritten in the 'integral' form $\mathbb{E}\big(\Phi(|Y_t|) - c_{p,q}|X_t|\big) \leq 0$, so it can be studied with the use of Burkholder's method described above. Combining these two steps and applying a certain homogenization argument (which involves dividing by $\|X_t\|_1$ when applying (2.7)), we will get the desired claim. Of course, it is absolutely not clear whether the functions $\Phi$, $\Psi$ can be chosen so that we obtain the best constant in (2.5); we will address this issue later.

Before we proceed, let us mention that (2.5) implies (1.6), but unfortunately, in the context of stochastic integrals only. Nevertheless, the proof of this weaker statement will provide us with certain auxiliary objects, to be needed later.

*Proof of* (1.6) *for stochastic integrals, assuming* (2.5). Let $X$ be an $\mathcal{H}$-valued martingale and let $Y$ be the stochastic integral, with respect to $X$, of some predictable process $H$ with values in $[-1,1]$. Let $A$ be an arbitrary event of positive probability. For $x \in \mathcal{H}$, let $x'$ be the 'sign' of $x$, given by $x' = x/|x|$ if $x \in \mathcal{H}\setminus\{0\}$ and $0' = 0$. We have

$$\int_A |Y_t|\mathrm{d}\mathbb{P} = \mathbb{E}\langle Y_t, 1_A Y_t'\rangle = \mathbb{E}\langle Y_t, \xi\rangle.$$

Let $(\xi_t)_{t\geq 0}$ be the $\mathcal{H}$-valued martingale induced by $\xi$: that is, put $\xi_t = \mathbb{E}(\xi|\mathcal{F}_t)$ for $t \geq 0$. By the properties of stochastic integrals, we obtain

$$\int_E |Y_t|\mathrm{d}\mathbb{P} = \mathbb{E}\langle Y_t, \xi\rangle = \mathbb{E}\int_0^t \mathrm{d}[Y,\xi]_s = \mathbb{E}\int_0^t H_s \mathrm{d}[X,\xi]_s = \mathbb{E}\int_0^t \mathrm{d}[X,\zeta]_s = \mathbb{E}\langle X_t, \zeta_t\rangle,$$

where $\zeta = (\zeta_t)_{t\geq 0}$ is the stochastic integral of $H$ with respect to $\xi$. But $\xi$ is bounded by 1; therefore, by Hardy–Littlewood–Polyá inequality, Hölder's inequality and (2.5) (applied to the martingales $\xi$, $\zeta$ and the exponents $p' > q' \geq 2$), we obtain

$$\begin{aligned}\mathbb{E}\langle X_t, \zeta_t\rangle &\leq \int_0^1 X_t^*(s)\zeta_t^*(s)\mathrm{d}s \\ &\leq \left(\int_0^1 (s^{1/p}X_t^*(s))^q \frac{\mathrm{d}s}{s}\right)^{1/q} \left(\int_0^1 (s^{1/p'}\zeta_t^*(s))^{q'} \frac{\mathrm{d}s}{s}\right)^{1/q'} \\ &= \left(\int_0^1 (s^{1/p'}\zeta_t^*(s))^{q'} \frac{\mathrm{d}s}{s}\right)^{1/q'} \|X_t\|_{p,q} \\ &\leq C_{p,q}\|\xi_t\|_1^{1/p'}\|X_t\|_{p,q}.\end{aligned} \tag{2.9}$$

However, we have $\|\xi_t\|_1 \leq \|\xi\|_1 = \mathbb{P}(A)$; putting all the above facts together, we get

$$\frac{1}{\mathbb{P}(A)^{1/p'}}\int_A |Y_t|\mathrm{d}\mathbb{P} \leq C_{p,q}\|X_t\|_{p,q},$$

and hence (1.6) follows, since $A$ was arbitrary. □

2.3. **The desired $L^{p,q} \to L^{p,\infty}$ inequality.** The proof of the estimate (1.6), under the general differential subordination, will rest on a somewhat similar two-step procedure. We establish first the intermediate sharp bound

$$\mathbb{E}(|Y_t| - \alpha)_+ \leq \mathbb{E}\Theta(|X_t|), \tag{2.10}$$

where $\alpha$ is a certain positive constant and $\Theta$ is an appropriate Young function. Observe that this estimate is of 'integral' form (2.1) and hence Burkholder's method applies here. The second step is to relate $\mathbb{E}\Theta(|X_t|)$ to the Lorentz norm $\|X_t\|_{p,q}$. This will be handled by means of the pointwise ('Young-type') estimate

$$\Theta(u) \leq \frac{u^q v}{q} + \rho(v), \tag{2.11}$$

for all $u$, $v \geq 0$ and some function $\rho$ on $[0,\infty)$. The application of this inequality to $u = X_t^*(s)$, $v = s^{q/p-1}$ and integrating over $s \in (0,1]$ gives

$$\mathbb{E}\Theta(|X_t|) = \int_0^1 \Theta(X_t^*(s))\mathrm{d}s \leq \frac{1}{q}\|X_t\|_{p,q}^q + \int_0^1 \rho(s^{q/p-1})\mathrm{d}s. \tag{2.12}$$

Combining this with (2.10) and using some additional homogenization/optimization arguments (similar to those above, leading to $L^p \to L^{p,\infty}$ estimates), will yield the desired assertion.

Before we proceed to the rigorous proof, let us comment that the special Young functions $\Phi$, $\Psi$ and $\Theta$ used above, as well as other parameters involved, will be given by quite complicated formulas. There is a natural question how to search for these objects; this will be discussed in detail in Remarks 3.7 and 4.6 below. Roughly speaking, the idea is as follows. First, one *guesses* the distributions of the extremal martingales in (1.6) and (2.5) (i.e., those $X$, $Y$, for which equalities hold); second, one construct $\Phi$, $\Psi$ and $\Theta$ so that equalities hold in the intermediate bounds (2.7), (2.8), (2.10) and (2.12).

## 3. A dual estimate

The purpose of this section is to prove the following fact.

**Theorem 3.1.** *Suppose that $X$, $Y$ are $\mathcal{H}$-valued martingales such that $Y$ is differentially subordinate to $X$. Then for any $p > q \geq 2$ the estimate* (2.5) *holds.*

As we have discussed in the previous section, the proof will be carried out in two steps. Consider Young functions $\Phi = \Phi_{p,q}$, $\Psi = \Psi_{p,q} : [0, \infty) \to [0, \infty)$ given by

$$\Phi(s) = 2^{1-q/p} \int_0^{s^q} \exp\left(\frac{p-q}{p} u^{1/q}\right) \mathrm{d}u$$

and

$$\Psi(t) = \begin{cases} (-\ln(2t))^q t^{q/p-1} - \Phi(-\ln(2t)) & \text{if } t < 1/2, \\ 0 & \text{if } t \geq 1/2. \end{cases}$$

The first step is to establish the Young-type inequality (2.6). This is straightforward, but we include the proof for the sake of completeness.

**Lemma 3.2.** *For $u \geq 0$ and $v > 0$, the estimate* (2.6)*, and hence also* (2.7)*, is valid.*

*Proof.* Fix $v > 0$, substitute $r = u^q$ and consider the function $F : [0, \infty) \to \mathbb{R}$, given by

$$F(r) = rv^{q/p-1} - \Phi(r^{1/q}) - \Psi(v).$$

We compute that $F'(r) = v^{q/p-1} - 2^{1-q/p} \exp\big((p-q)r^{1/q}/p\big)$. Therefore, if $v \geq 1/2$, then $F$ is decreasing and $F(r) \leq F(0) = 0$. On the other hand, if $v < 1/2$, then $F$ attains its maximal value at $r = (-\ln(2v))^q$: so, $F(r) \leq F((-\ln(2v))^q) = 0$. □

The main technical difficulty lies in the proof of the second step, the inequality (2.8). Consider the constant

$$c_{p,q} = \frac{1}{2} \int_0^\infty \Phi(t) e^{-t} \mathrm{d}t = 2^{-q/p} \left(\frac{p}{q}\right)^q \Gamma(q+1) = \frac{q}{p} C_{p',q'}^q. \tag{3.1}$$

We will prove the following fact.

**Theorem 3.3.** *Let $t \geq 0$ be a fixed parameter and suppose that $X$, $Y$ are $\mathcal{H}$-valued martingales such that $\|X_t\|_\infty \leq 1$. Then* (2.8) *is true.*

We will apply Burkholder's method, described in the previous section. Consider the domain $D = \{(x, y) \in \mathcal{H} \times \mathcal{H} : |x| \leq 1\}$ and let

$$D_1 = \{(x, y) \in D : |x| + |y| < 1\}, \qquad D_2 = \{(x, y) \in D : |x| + |y| > 1\}.$$

The estimate (2.8) can be rewritten in the form (2.1), with $V : D \to \mathbb{R}$ given by $V(x, y) = \Phi(|y|) - c_{p,q}|x|$. Define the special function $U : D \to \mathbb{R}$ by

$$U(x,y) = \begin{cases} c_{p,q}(|y|^2 - |x|^2) & \text{if } (x,y) \in \overline{D_1}, \\ |x|\Phi\big(|x| + |y| - 1\big) + (1 - |x|)e^{|x|+|y|} \int_{|x|+|y|}^\infty \Phi(t-1)e^{-t}\mathrm{d}t - c_{p,q} & \text{if } (x,y) \in \overline{D_2}. \end{cases}$$

We will check that $U$ enjoys the required properties 1°, 2° and 3°. We start with the observation that the function is of class $C^1$, and proceed to the proof of the condition 3°.

**Lemma 3.4.** *The estimate* (2.3) *holds with*

$$c(x,y)=\begin{cases}2c_{p,q} & \text{if } (x,y)\in D_1,\\ \displaystyle\int_0^\infty \Phi''(t+|x|+|y|-1)e^{-t}\,dt & \text{if } (x,y)\in D_2.\end{cases}$$

*The function $c$ is nonnegative and satisfies the boundedness condition* (2.2).

*Proof.* If $(x,y)\in D_1$, then the estimate (2.3) is evident: actually, both sides are equal. If $(x,y)\in D_2$, then some lengthy, but rather straightforward calculations reveal that the left-hand side of (2.3) equals $I_1+I_2+I_3$, where

$$\begin{aligned}I_1&=-\int_0^\infty \Phi'''(t+u)e^{-t}\mathrm{d}t\cdot(\langle x',h\rangle+\langle y',k\rangle)^2|x|,\\ I_2&=\left[\Phi'(u)-u\Phi''(u)-u\int_0^\infty \Phi'''(t+u)e^{-t}\mathrm{d}t\right]\frac{(|k|^2-\langle y',k\rangle^2)}{|y|},\\ I_3&=\int_0^\infty \Phi''(t+u)e^{-t}\mathrm{d}t\cdot(|k|^2-|h|^2),\end{aligned}$$

where $u=|x|+|y|-1$. Here, as before, we use the notation $x'=x/|x|$ for $x\neq 0$, and $x'=0$ otherwise. Observe that both terms $I_1$ and $I_2$ are nonpositive. Indeed, this follows at once from the fact that the third derivative

$$\Phi'''(s)=q2^{1-q/p}\exp\left(\frac{p-q}{p}s\right)s^{q-3}\left[\left(\frac{p-q}{p}\right)^2s^2+2\left(\frac{p-q}{p}\right)(q-1)s+(q-1)(q-2)\right]$$

is nonnegative (then by the mean-value property we have $\Phi'(u)-u\Phi''(u)\leq 0$). Here is the place where we use the assumption $q\geq 2$. This establishes the estimate (2.3). The nonnegativity of $c$ and the boundedness condition (2.2) are evident. The proof is complete. □

The next step is to establish the size requirements.

**Lemma 3.5.** *The conditions 1° and 2° are satisfied.*

*Proof.* The estimate 1° depends on $x$ and $y$ through their norms only, so we may assume that $x,\,y\in\mathbb{R}$. By the previous lemma, the function $F:t\mapsto U(tx,ty)$ is concave on $\{t\in\mathbb{R}:|tx|\leq 1\}$, it is also even there. This implies $F(1)\leq F(0)=0$, which is exactly the first condition. The proof of 2° will be more involved. As previously, we may assume that $\mathcal{H}=\mathbb{R}$, actually, we may even restrict ourselves to nonnegative $x$ and $y$. By the previous lemma, the function $t\mapsto U(tx,y)$ is concave on the set $\{t\in[0,\infty):tx\leq 1\}$ and hence it is enough to establish the majorization for $x\in\{0,1\}$. If $x=1$ then both sides are equal, so let us assume that $x=0$. Consider the auxiliary function $F(y)=U(0,y)-\Phi(y)$ for $y\geq 0$. Note that $F(0)=F'(0)=0$, $F''(0)=2c_{p,q}>0$ and $F'''(y)=-\Phi'''(y)<0$ for $y\in(0,1)$, which, by the continuity of $F$, implies that it is enough to prove $F(y)\geq 0$ on $[1,\infty)$. To this end, we need more calculations. If $y\in(1,\infty)$, then we have the identity

$$F(y)=e^y\left[\int_y^\infty \Phi(t-1)e^{-t}\mathrm{d}t-c_{p,q}e^{-y}-\Phi(y)e^{-y}\right]=e^yJ(y).$$

A direct differentiation shows that

$$J'(y)=e^{-y}\Big[\Phi(y)-\Phi(y-1)-\Phi'(y)+c_{p,q}\Big].$$

Denote the expression in the square brackets by $K(y)$ and note that

$$K'(y) = \Big(\Phi'(y) - \Phi'(y-1) - \Phi''(y)\Big) \leq 0,$$

where the latter bound is due to the mean-value theorem (since $\Phi''' \geq 0$). Furthermore, as we will prove now, $K$ is negative for large $y$. Indeed,

$$\begin{aligned}\Phi(y) - \Phi(y-1) &= 2^{1-q/p} q \int_{y-1}^{y} \exp\left(\frac{p-q}{p}s\right) s^{q-1} \mathrm{d}s \\ &\leq 2^{1-q/p} \exp\left(\frac{p-q}{p}y\right) \cdot (y^q - (y-1)^q),\end{aligned}$$

which implies

$$\Phi(y) - \Phi(y-1) - \Phi'(y) \leq 2^{1-q/p} \exp\left(\frac{p-q}{p}y\right)(y^q - (y-1)^q - qy^{q-1}).$$

The latter expression tends to $-\infty$ as $y \to \infty$, so indeed $K(y) \leq 0$ for $y$ big enough. Now, observe that $F(1) = c_{p,q} - \Phi(1)$. If this quantity were negative (equivalently: $c_{p,q} < \Phi(1)$), then we would also have $J(1) < 0$ and

$$K(1) = \Phi(1) - \Phi'(1) + c_{p,q} < 2\Phi(1) - \Phi'(1) \leq 0.$$

To see the latter estimate, note that the function $\lambda \mapsto 2\Phi(\lambda) - \lambda\Phi'(\lambda)$ is concave (since $\Phi''' \geq 0$) and it vanishes, along with its derivative, at $\lambda = 0$. But $K$ is decreasing, as we proved above: therefore, we would have $K < 0$ and hence also $J < J(1) < 0$ on $(1, \infty)$. But this is a contradiction: we have $J(y) \to 0$ as $y \to \infty$. This proves that $F(1) \geq 0$ and $J(1) \geq 0$. Since $K$ is decreasing on $(1, \infty)$, it is either negative there, or it changes its sign once from plus to minus. Therefore, $J$ is either decreasing, or there is $y_0 \in (1, \infty)$ such that $J$ is increasing on $(1, y_0)$ and decreasing on $(y_0, \infty)$. Since $J(1) \geq 0$ and $J(y) \to 0$ as $y \to \infty$, this proves that $J$, and hence also $F$, are nonnegative on $[1, \infty)$. This is the desired claim. □

We are ready for the proof of the auxiliary $\Phi$-estimate.

*Proof of Theorem 3.3.* Fix arbitrary $X, Y$ as in the statement and an auxiliary parameter $r \in (0,1)$. By Lemma 2.1, applied to the dilated pair $(rX, rY)$, there is a sequence $(\tau_n)_{n\geq 1}$ of stopping times increasing to infinity such that

$$\mathbb{E}U(rX_{\tau_n\wedge t}, rY_{\tau_n\wedge t}) \leq \mathbb{E}U(rX_0, rY_0).$$

Therefore, the application of 1° and 2° yields the estimate $\mathbb{E}V(rX_{\tau_n\wedge t}, rY_{\tau_n\wedge t}) \leq 0$, or

$$\mathbb{E}\Phi(|rY_{\tau_n\wedge t}|) \leq c_{p,q}\mathbb{E}|rX_{\tau_n\wedge t}| \leq c_{p,q}\mathbb{E}|X_{\tau_n\wedge t}|.$$

Letting $r \to 1$ and $n \to \infty$ gives the estimate (2.8), by means of Fatou's lemma. The sharpness follows from the fact that the constant $c_{p,q}$ is already the best possible in the weaker inequality $\mathbb{E}\Phi(|Y_t|) \leq c_{p,q}$. See Theorem 6.1 in [11] for details. □

We proceed to the proof of the dual Lorentz-norm estimate.

*Proof of* (2.5). By homogeneity, we may and do assume that $\|X_t\|_\infty \leq 1$. The estimate (2.6), with $u = X^*(s)$ and $v = s/\|X_t\|_1$, combined with (2.8), gives

$$
\begin{aligned}
\int_0^1 (Y_t^*(s))^q s^{q/p-1}\mathrm{d}s &= \|X_t\|_1^{q/p-1}\int_0^1 (Y_t^*)^q\left(s/\|X_t\|_1\right)^{q/p-1}\mathrm{d}s\\
&\leq \|X_t\|_1^{q/p-1}\left[\int_0^1 \Phi\big(Y_t^*(s)\big)\mathrm{d}s + \int_0^1 \Psi\big(s/\|X_t\|_1\big)\mathrm{d}s\right]\\
&= \|X_t\|_1^{q/p-1}\left[\mathbb{E}\Phi(|Y_t|) + \|X_t\|_1\int_0^{\|X_t\|_1^{-1}}\Psi(s)\mathrm{d}s\right]\\
&\leq \|X_t\|_1^{q/p}\left[c_{p,q} + \int_0^{1/2}\Psi(s)\mathrm{d}s\right] = C_{p',q'}^q\|X_t\|_1^{q/p}.
\end{aligned}
\tag{3.2}
$$

This is the desired estimate. □

As we have seen in Section 2, the estimate (2.5) yields (1.6) for stochastic integrals. We will now show that the latter result is sharp, thus proving that the constant in (2.5) is also the best possible. It will be convenient for us to work in the discrete-time case; this context will also be useful in our further considerations on Fourier multipliers.

*Sharpness of* (1.6) *for martingale transforms.* Fix $p \leq q \leq 2$. We will construct an appropriate martingale example which attains the constant $C_{p,q}$ in the limit. Let $\kappa : (0,\infty) \to [0,\infty)$ be a non-increasing function given by

$$\kappa(t) = (-\ln(2t))^{1/(q-1)} t^{q'/p'-1}\chi_{(0,1/2]}(t). \tag{3.3}$$

Pick $\alpha \in (0,1)$ and let $f = f^{(\alpha)}$ be the discretized version of $\kappa$, defined by $f(t) = \kappa(\alpha^{n-1}/2)$ if $t \in (\alpha^n/2, \alpha^{n-1}/2]$ and $f(t) = 0$ for $t \in (1/2, 1]$. Note that $f$ is non-increasing and $f \leq \kappa$, so

$$\|f\|_{p,q} \leq \|\kappa\|_{p,q} = \left(\int_0^{1/2}(-\ln(2t))^q t^{q/p-1}\mathrm{d}t\right)^{1/q} = C_{p,q}^{q-1}.$$

Consider the probability space equal to $((0,1], \mathcal{B}(0,1), |\cdot|)$, the interval $(0,1]$ with its Borel subsets and the Lebesgue measure, equipped with the following filtration: $\mathcal{F}_0$ is the trivial $\sigma$-field, while for $n \geq 1$, $\mathcal{F}_n$ is generated by the intervals $(0, \alpha^{n-1}/2]$, $(\alpha^{n-1}/2, \alpha^{n-2}/2]$, $(\alpha^{n-2}/2, \alpha^{n-3}/2]$, …, $(\alpha/2, 1/2]$ and $(1/2, 1]$. Next, for any $n = 0, 1, 2, \ldots$, let $f_n = \mathbb{E}(f|\mathcal{F}_n)$ and define $g = (g_n)_{n\geq 0}$ as the transform of $(f_n)_{n\geq 0}$, determined by $g_0 = f_0$ and $dg_n = -\operatorname{sgn}(g_{n-1})df_n$ (with the convention $\operatorname{sgn}(0) = 1$). Note that

$$\lim_{n\to\infty}\|f_n\|_{p,q} = \|f\|_{p,q} \leq C_{p,q}^{q-1}. \tag{3.4}$$

Furthermore, as we have already mentioned above, the function $f$ is non-increasing: this implies that for $n \geq 1$ and $t \in (\alpha^n/2, \alpha^{n-1}/2]$ we have

$$f_n(t) = \frac{2}{\alpha^{n-1}}\int_0^{\alpha^{n-1}/2} f\mathrm{d}x \geq f(t) = f_{n+1}(t) = f_{n+2}(t) = \ldots,$$

that is, $df_{n+1}(t) \leq 0$ and $df_{n+2}(t) = df_{n+3}(t) = \ldots = 0$. By the choice of the transforming sequence, we obtain that $g_\infty$, the almost sure limit of $(g_n)_{n\geq 0}$, satisfies

$$|g_\infty(t)| = |g_{n+1}(t)| = |g_n(t)| + |df_{n+1}(t)| \geq -df_{n+1}(t) = f_n(t) - f_{n+1}(t) = f_n(t) - f(t).$$

Therefore,

$$\begin{aligned}\int_\Omega |g_\infty| \mathrm{d}\mathbb{P} &= \int_0^{1/2} |g_\infty(t)|\mathrm{d}t + \int_{1/2}^1 |g_\infty(t)|\mathrm{d}t\\ &= \sum_{n\geq 1}\int_{\alpha^n/2}^{\alpha^{n-1}/2} |g_\infty(t)|\mathrm{d}t + \int_{1/2}^1 2f_0 \mathrm{d}t\\ &\geq \sum_{n\geq 1}\int_{\alpha^n/2}^{\alpha^{n-1}/2} \big(f_n(t)-f(t)\big)\mathrm{d}t + \int_0^{1/2} f(t)\mathrm{d}t\\ &= \sum_{n\geq 1}\int_{\alpha^n/2}^{\alpha^{n-1}/2} f_n(t)\mathrm{d}t = \int_0^{1/2}\sum_{n\geq 1} f_n(t)\chi_{(\alpha^n/2,\alpha^{n-1}/2]}(t)\mathrm{d}t.\end{aligned}$$

Now, observe that as $\alpha\uparrow 1$, we have the pointwise convergence

$$\sum_{n\geq 1} f_n(t)\chi_{(\alpha^n/2,\alpha^{n-1}/2]}(t) \to \frac{1}{t}\int_0^t \kappa(u)\mathrm{d}u$$

and hence, by Fatou's lemma,

$$\liminf_{\alpha\uparrow 1}\int_\Omega |g_\infty|\mathrm{d}\mathbb{P} \geq \int_0^{1/2}\frac{1}{t}\int_0^t \kappa(u)\mathrm{d}u\mathrm{d}t = \int_0^{1/2}(-\ln(2t))^q t^{q/p-1}\mathrm{d}t = C_{p,q}^q.$$

Combining this with (3.4), we see that with a proper choice of $n$ and $\alpha$, the ratio $\int_\Omega |g_n|\mathrm{d}\mathbb{P}/\|f_n\|_{p,q}$, and hence also $\|g_n\|_{p,\infty}/\|f_n\|_{p,q}$, can be made bigger than $C_{p,q}-\varepsilon$. This gives the desired sharpness. □

*Remark* 3.6. We conclude with a remark which will be useful later in Section 5, during the study of Fourier multipliers. Namely, by a simple symmetrization, we may assume that the extremal martingales constructed above have expectation zero. Suppose that $\alpha\in(0,1)$ is given and $(f,g)$ is the corresponding pair of martingales on $(0,1]$. We consider the new probability space $([-1,1],\mathcal{B}([-1,1]),|\cdot|/2)$ and for any $n\geq 0$, define

$$\tilde{f}_n(\omega) = \begin{cases} f_n(\omega) & \text{if } \omega\in(0,1],\\ -f_n(-\omega) & \text{if } \omega\in[-1,0),\end{cases} \qquad \tilde{g}_n(\omega) = \begin{cases} g_n(\omega) & \text{if } \omega\in(0,1],\\ -g_n(-\omega) & \text{if } \omega\in[-1,0)\end{cases}$$

(the values for $\omega=0$ are irrelevant). Finally, we set $f_{-1}=g_{-1}=0$. Then $f=(f_n)_{n\geq -1}$, $g=(g_n)_{n\geq -1}$ are martingales relative to natural filtration, they have expectation zero, $g$ is the transform of $f$ by a predictable sequence with values in $\{-1,1\}$ and the distributions of $(|f_n|,|g_n|)$ and $(|\tilde{f}_n|,|\tilde{g}_n|)$ coincide for each $n$. Hence, by a proper choice of $n$ and $\alpha$, we may make the ratio $\|\tilde{g}_n\|_{p,\infty}/\|\tilde{f}_n\|_{p,q}$ arbitrarily close to $C_{p,q}$.

*Remark* 3.7. Now we present the details on the search for $\Phi$ and $\Psi$, which was outlined in the previous section. Suppose that there is a nontrivial pair $(X,Y)$ for which both sides of (2.5) equal. Then equality holds in (2.7) and (2.8) as well; let us take a look at the first bound. Equality holds here, so both sides of (2.6), with $u=Y_t^*(s)$ and $v=s$, should also be the same, for all $s$. This suggests to inspect for which $u$, $v$ the estimate (2.6) becomes an equality. As we have seen in the proof of Lemma 3.2, this holds if $qu^{q-1}v^{q/p-1}=\Phi'(u)$, and coming back to the function $Y_t^*$, we obtain

$$qY_t^*(s)^{q-1}s^{q/p-1} = \Phi'(Y_t^*(s)). \tag{3.5}$$

Therefore, if we knew the explicit formula for the non-increasing rearrangement $Y_t^*$, this would lead us to the derivative of $\Phi$ and hence also to the function $\Phi$ itself; the formula for the second function, $\Psi$, would then be immediate and would follow from the fact that equality holds in (2.6) for $u = Y_t^*(s)$ and $v = s$.

So, all we need is the explicit formula for $Y_t^*$; here the second inequality (2.8) comes into play. In [11] (see Theorem 6.1 there), Burkholder studied a wide class of general sharp inequalities of the form $\mathbb{E}\Phi(|Y_t|) \leq C_\Phi$, $t \geq 0$, under the requirement $\|X_t\|_\infty \leq 1$ and certain mild growth assumptions of $\Phi$. Although these inequalities are a little weaker than what we need - the term $\|X_t\|_1$ is not present on the right - we may still take a look at the extremal martingale pairs constructed by Burkholder. It can be verified readily that for such processes, we have $Y_t^*(s) = (-\ln(2s))_+$. Plugging this into (3.5), we obtain the functions $\Phi$ and $\Psi$ defined at the beginning of this section.

We should point out here that the assumption $p \geq q \geq 2$ is essential here. Without this condition, we may still perform the above procedure and obtain the function $\Phi$ given by the same formula, but then the estimate (2.8) fails. In other words, either the inequality (2.5) cannot be proven with the splitting into two intermediate bounds, or our guess for the extremal $Y$ was incorrect.

## 4. Proof of Theorem 1.2 for general martingales

We start with the introduction of some auxiliary objects. Let $\kappa : (0, 1/2] \to [0, \infty)$ be the function given by (3.3). The restriction to the domain $(0, 1/2]$ guarantees that the function $\kappa$ is strictly decreasing, hence invertible: let $\kappa^{-1} : [0, \infty) \to (0, 1/2]$ be the inverse. Define $\Theta : [0, \infty) \to [0, \infty)$ by the formula

$$\Theta(t) = -\int_0^t \ln(2\kappa^{-1}(s))\mathrm{d}s.$$

Since $\kappa$ is decreasing, so is $\kappa^{-1}$ and hence $\Theta$ is a convex function. Some steps leading to this somewhat strange object are described in Remark 4.6 below.

As we discussed in Section 2, the proof of the Lorentz-norm inequality (1.6) will be based on two-step procedure, exploiting the estimates (2.10) and (2.11). Let us start with the second bound, whose analysis is straightforward.

**Lemma 4.1.** *For any $u \geq 0$ and $v > 0$ we have the inequality*

$$\Theta(u) - \frac{u^q v}{q} \leq \Theta\left(\left(-\frac{\ln(2v^{p/(q-p)})}{v}\right)_+^{1/(q-1)}\right) - \frac{v}{q}\left(-\frac{\ln(2v^{p/(q-p)})}{v}\right)_+^{q/(q-1)}. \tag{4.1}$$

*Proof.* Fix $v > 0$ and denote the left-hand side by $F(u)$. By the definition of $\Theta$, we have

$$F'(u) = \Theta'(u) - u^{q-1}v = -\ln(2\kappa^{-1}(u)) - u^{q-1}v.$$

Therefore, $F' > 0$ if and only if $u > \kappa(\exp(-u^{q-1}v)/2)$, or $u < \left(-\ln(2v^{p/(q-p)})/v\right)_+^{1/(q-1)}$. This yields $F(u) \leq F\left(\left(-\ln(2v^{p/(q-p)})/v\right)_+^{1/(q-1)}\right)$, which is the claim. □

We turn our attention to the inequality (2.10). We will show the following statement.

**Theorem 4.2.** *Suppose that $p \leq q \leq 2$. Let $X$, $Y$ be two $\mathcal{H}$-valued martingales such that $Y$ is differentially subordinate to $X$. Then we have*

$$\mathbb{E}\left(|Y_t| - \frac{\gamma(0)}{2}\right)_+ \leq \mathbb{E}\Theta(|X_t|), \qquad t \geq 0. \tag{4.2}$$

This result will be proved by Burkholder's method. Let $D = \mathcal{H} \times \mathcal{H}$. The estimate (4.2) is of the form (2.1), with $V : D \to \mathbb{R}$ given by $V(x,y) = (|y| - \gamma(0)/2)_+ - \Theta(|x|)$. To introduce the corresponding special function $U$, we need an additional object. Let $\gamma : [0,\infty) \to [0,\infty)$ be defined by

$$\gamma(t) = \exp\big(\Theta'(t)\big) \int_t^\infty \exp\big(-\Theta'(s)\big)\mathrm{d}s. \tag{4.3}$$

Let us prove some basic facts about $\gamma$, which will be needed later.

**Lemma 4.3.** *The function $\gamma$ is increasing and satisfies the differential equation*

$$\frac{1+\gamma'(t)}{\gamma(t)} = \Theta''(t), \qquad t > 0. \tag{4.4}$$

*Furthermore, for any $t \geq 0$ we have*

$$t \leq \Theta'(t)\gamma(t). \tag{4.5}$$

*Proof.* Since $\kappa$ is decreasing, the monotonicity of $\gamma$ will follow if we show that the function

$$u \mapsto \gamma(\kappa(u)) = -\frac{1}{u}\int_0^u s\kappa'(s)\mathrm{d}s \tag{4.6}$$

is decreasing. This, in turn, will be established if we prove that $s\kappa'(s)$ increases as $s$ increases. However, we compute that

$$(s\kappa'(s))' = (-\ln(2s))^{\frac{1}{q-1}-2} s^{\frac{q'}{p'}-2}\left[\frac{2-q}{q-1} + \frac{2}{q-1}\left(\frac{q'}{p'}-1\right)\ln(2s) + \left(\frac{q'}{p'}-1\right)^2 (\ln(2s))^2\right]$$

is positive, since so are the three terms in the square brackets. (Here is the place where the assumption $q \leq 2$ plays a role: for $q > 2$, the above expression becomes negative for $s$ close to $1/2$.) The equation (4.4) follows by a direct differentiation of (4.3). Finally, to show (4.5), note that both sides are equal for $t = 0$, and by the previous two properties,

$$\left(\frac{t}{\gamma(t)} - \Theta'(t)\right)' = \frac{1}{\gamma(t)} - \frac{t\gamma'(t)}{\gamma(t)^2} - \Theta''(t) = -\frac{t\gamma'(t)}{\gamma(t)^2} - \frac{\gamma'(t)}{\gamma(t)} \leq 0.$$

The proof is complete. $\square$

Now we may introduce the special function $U$. Recall that $D = \mathcal{H} \times \mathcal{H}$ and consider the subdomains

$$\begin{aligned} D_1 &= \{(x,y) \in D : |x| + |y| < \gamma(0)\}, \\ D_2 &= \{(x,y) \in D : |y| > \gamma(|x|)\}, \\ D_3 &= \{(x,y) \in D : \gamma(0) - |x| < |y| < \gamma(|x|)\}. \end{aligned}$$

The function $U$ is given by

$$U(x,y) = \begin{cases} \dfrac{|y|^2 - |x|^2}{2\gamma(0)} & \text{if } (x,y) \in \overline{D_1}, \\ |y| - \Theta(|x|) - \dfrac{\gamma(0)}{2} & \text{if } (x,y) \in \overline{D_2}, \\ \gamma(s) - t - \Theta(s) - \Theta'(s)t - \dfrac{\gamma(0)}{2} & \text{if } (x,y) \in D_3. \end{cases}$$

Here in the definition of $U(x,y)$ for $(x,y) \in D_3$, the letters $s$, $t$ denote the unique positive numbers such that $|x| = s + t$ and $|y| = \gamma(s) - t$. The formula for $U$ on the domain

$D_3$ might look a little intricate, but it has a very nice geometric interpretation, which is conveniently explained in the case $\mathcal{H} = \mathbb{R}$. Namely, given $(x, y) \in D_3$ with $x$, $y \geq 0$, we draw a line segment of slope $-1$ which joins $(x, y)$ with a point at the boundary $\partial D_3$ (which is precisely $(s, \gamma(s))$). Then we require that $U$ is linear along this segment, and that $U$ is of class $C^1$; this yields the above formula.

Now we will check that $U$ satisfies the appropriate conditions. As previously, we start with the concavity property 3°.

**Lemma 4.4.** *The function $U$ satisfies* (2.3) *with*

$$c(x,y) = \begin{cases} (\gamma(0))^{-1} & \text{if } (x,y) \in D_1, \\ (\gamma(|x|))^{-1} & \text{if } (x,y) \in D_2, \\ (\gamma(s))^{-1} & \text{if } (x,y) \in D_3. \end{cases}$$

*(On $D_3$, the number $s \geq 0$ is uniquely determined by the requirements $s + t = |x|$ and $\gamma(s) - t = |y|$). The function $c$ is nonnegative and satisfies the condition* (2.2).

*Proof.* The second and the third part of the lemma is evident, so we may focus on the inequality (2.3). If $(x, y) \in D_1$, then both sides of this estimate are equal. If $(x, y) \in D_2$, then the bound becomes

$$\frac{|k|^2 - \langle y', k\rangle^2}{|y|} - \Theta''(|x|)\langle x', h\rangle^2 - \Theta'(|x|) \cdot \frac{|h|^2 - \langle x', h\rangle^2}{|x|} \leq \frac{|k|^2 - |h|^2}{\gamma(|x|)}.$$

However, we have $(|k|^2 - \langle y', k\rangle^2)/|y| \leq |k|^2/|y| \leq |k|^2/\gamma(|x|)$. Furthermore, we have $\Theta'(|x|)/|x| \geq 1/\gamma(|x|)$, so

$$-\left(\Theta'(|x|) - \frac{1}{\gamma(|x|)}\right)|h|^2 \leq -\left(\Theta'(|x|) - \frac{1}{\gamma(|x|)}\right)\langle x', h\rangle^2.$$

Finally, we have $-\Theta''(|x|) + 1/\gamma(|x|) = -\gamma'(|x|)/\gamma(|x|) \leq 0$, so

$$(-\Theta''(|x|) + 1/\gamma(|x|))\,\langle x', h\rangle^2 \leq 0.$$

Summing the three displayed inequalities above, we get (2.3). If $(x, y) \in D_3$, then the analysis is a little more involved. Let us, for a moment, use the notation $U^{\mathcal{H}}$ to indicate the Hilbert space we work in. Then $U^{\mathcal{H}}(x, y) = U^{\mathbb{R}}(|x|, |y|)$ and the inequality (2.3) is equivalent to

$$\begin{aligned} &U^{\mathbb{R}}_{xx}(p)\langle x', h\rangle^2 + 2U^{\mathbb{R}}_{xy}(p)\langle x', h\rangle\langle y', k\rangle + U^{\mathbb{R}}_{yy}(p)\langle y', k\rangle^2 \\ &\qquad + U^{\mathbb{R}}_{x}(p) \cdot \frac{|h|^2 - \langle x', h\rangle^2}{|x|} + U^{\mathbb{R}}_{y}(p) \cdot \frac{|k|^2 - \langle y', k\rangle^2}{|y|} \leq \frac{|k|^2 - |h|^2}{\gamma(s)}, \end{aligned} \tag{4.7}$$

where $p = (|x|, |y|) = (s + t, \gamma(s) - t)$. Recall that by the very definition of $U^{\mathbb{R}}$, we have

$$U^{\mathbb{R}}(s + t, \gamma(s) - t) = \gamma(s) - t - \Theta(s) - \Theta'(s)t. \tag{4.8}$$

Differentiating with respect to $s$ and with respect to $t$ gives the identities

$$\begin{aligned} U^{\mathbb{R}}_{y}(p) + U^{\mathbb{R}}_{y}(p)\gamma'(s) &= \gamma'(s) - \Theta'(s) - \Theta''(s), \\ U^{\mathbb{R}}_{x}(p) - U^{\mathbb{R}}_{y}(p) &= -1 - \Theta'(s). \end{aligned} \tag{4.9}$$

This yields

$$\begin{aligned} U_x^{\mathbb{R}}(p)(1+\gamma'(s)) &= -\Theta'(s)(1+\gamma'(s)) - \Theta''(s)t \\ &= -\Theta'(s)(1+\gamma'(s)) - \frac{1+\gamma'(s)}{\gamma(s)}\cdot t \leq -\frac{1+\gamma'(s)}{\gamma(s)}\cdot (s+t), \end{aligned} \tag{4.10}$$

where the latter bound follows from (4.5). By above inequality, we obtain

$$\left(\frac{U_x^{\mathbb{R}}(p)}{|x|}+\frac{1}{\gamma(s)}\right)|h|^2 \leq \left(\frac{U_x^{\mathbb{R}}(p)}{s+t}+\frac{1}{\gamma(s)}\right)\langle x',h\rangle^2,$$

so we may assume $|h| = \langle x', h\rangle$ in (4.7). Similarly, we may assume there that $|k| = \langle y', k\rangle$: by (4.9), we have

$$U_y^{\mathbb{R}}(p)(1+\gamma'(s)) = 1+\gamma'(s) - \Phi''(s)t = 1+\gamma'(s) - \frac{1+\gamma'(s)}{\gamma(s)}\cdot t. \tag{4.11}$$

Consequently, $U_y^{\mathbb{R}}(p)/(\gamma(s)-t) = 1/\gamma(s)$ and hence

$$\frac{U_y^{\mathbb{R}}(p)}{\gamma(s)-t}(|k|^2 - \langle y',k\rangle^2) - \frac{|k|^2}{\gamma(s)} = -\frac{\langle y',k\rangle^2}{\gamma(s)}.$$

Therefore, it is enough to prove that

$$U_{xx}^{\mathbb{R}}(p)\langle x',h\rangle^2 + 2U_{xy}^{\mathbb{R}}(p)\langle x',h\rangle\langle y',k\rangle + U_{yy}^{\mathbb{R}}(p)\langle y',k\rangle^2 \leq \frac{\langle y',k\rangle^2 - \langle x',h\rangle^2}{\gamma(s)}.$$

To do this, we differentiate the equations (4.10), (4.11) with respect to $s$ and with respect to $t$, and compute as above, to obtain

$$\begin{aligned} U_{xx}^{\mathbb{R}}(p)(1+\gamma'(s)) &= -\frac{1+\gamma'(s)}{\gamma(s)} - \frac{\gamma(s)-t}{\gamma^2(s)}\cdot\gamma'(s), \\ U_{xy}^{\mathbb{R}}(p)(1+\gamma'(s)) &= -\frac{\gamma(s)-t}{\gamma^2(s)}\cdot\gamma'(s), \\ U_{yy}^{\mathbb{R}}(p)(1+\gamma'(s)) &= \frac{1+\gamma'(s)}{\gamma(s)} - \frac{\gamma(s)-t}{\gamma^2(s)}\cdot\gamma'(s). \end{aligned}$$

This gives the desired estimate: we have

$$\begin{aligned} &U_{xx}^{\mathbb{R}}(p)\langle x',h\rangle^2 + 2U_{xy}^{\mathbb{R}}(p)\langle x',h\rangle\langle y',k\rangle + U_{yy}^{\mathbb{R}}(p)\langle y',k\rangle^2 \\ &= -\frac{\gamma(s)-t}{\gamma^2(s)(1+\gamma'(s))}\cdot\gamma'(s)(\langle x',h\rangle + \langle y',k\rangle)^2 + \frac{\langle y',k\rangle^2 - \langle x',h\rangle^2}{\gamma(s)} \leq \frac{\langle y',k\rangle^2 - \langle x',h\rangle^2}{\gamma(s)}. \end{aligned}$$

The proof is complete. □

Now, we will verify that $U$ satisfies the appropriate size requirements.

**Lemma 4.5.** *The conditions 1° and 2° hold true.*

*Proof.* The proof of 1° is the same as that in Lemma 3.5. In the proof of the majorization 2°, we may restrict ourselves to the case $\mathcal{H} = \mathbb{R}$ and nonnegative $x$, $y$. Fix such an $x$ and consider the function $F(y) = U(x,y) - (y-\gamma(0)/2)_+ + \Theta(x)$, $y \geq 0$. Observe that $F(y) = 0$ for $y \geq \gamma(x)$ (i.e., for $(x,y) \in \overline{D_2}$. If the reverse estimate holds, then $0 \leq U_y(x,y) \leq 1$: this is trivial if $(x,y) \in D_1$, and follows at once from (4.11) if $(x,y) \in D_3$ (indeed: we have $U_y(s+t,\gamma(s)-t) = 1 - t/\gamma(s)$). Putting all the above facts together, we see that the function $F$ is increasing on $[0,\gamma(0)/2]$, decreasing on $[\gamma(0)/2,\gamma(x)]$ and vanishes for $y\geq\gamma(x)$. Consequently, it is enough to prove that $F(0)\geq 0$, i.e.,

$$G(x):=U(x,0)+\Theta(x)\geq 0$$

for all $x\geq 0$. Let us study the properties of $G$. First, note that we have $G(0)=G'(0)=0$ and $G''(x)=\Theta''(x)-(\gamma(0))^{-1}$ for $x\leq\gamma(0)/2$. However, by the very definition of $\Theta$, we have $\Theta'(\kappa(u))=-\ln(2u)$ and hence $\Theta''(\kappa(u))\cdot(-u\kappa'(u))=1$ by the direct differentiation. If $u\to 1/2$, then $\kappa(u)$ and $\kappa'(u)$ tend to zero. Furthermore, as we have proved in Lemma 4.3, the nonnegative function $u\mapsto -u\kappa'(u)$ is decreasing. Since $\kappa$ is also decreasing (on $(0,1/2)$), we conclude that $G''$ is decreasing on $(0,\gamma(0)/2)$ and $G''(x)\to\infty$ as $x\to 0$. Therefore, there are scenarios possible on $[0,\gamma(0)/2]$: either $G$ is convex there, or there is $x_0\in(0,\gamma(0)/2)$ such that $G$ is convex on $[0,x_0]$ and concave $[x_0,\gamma(0)/2]$. No matter which case happens, there is no $x\in(0,\gamma(0)/2]$ for which $G'(x)=0$ and $G(x)<0$.

Next, we turn our attention to the analysis of $G$ on $[\gamma(0)/2,\infty)$. We rewrite the estimate $G>0$ in the equivalent form

$$\overline{G}(s)=\Theta(s+\gamma(s))-\Theta(s)-\Theta'(s)\gamma(s)-\frac{\gamma(0)}{2}\geq 0,$$

for $s\geq 0$. Note that by a direct differentiation,

$$-u\kappa'(u)=\left[-\frac{1}{(q-1)\ln(2u)}+1-\frac{q'}{p'}\right]\kappa(u),$$

so if $u$ is close to 0, then $-u\kappa'(u)\sim\kappa(u)$. Precisely, if $u\in(0,(2e)^{-1}]$, then

$$\left(1-\frac{q'}{p'}\right)\kappa(u)\leq -u\kappa'(u)\leq\left(\frac{1}{q-1}+1-\frac{q'}{p'}\right)\kappa(u). \tag{4.12}$$

Consequently,

$$\Theta''(\kappa(u))=-\frac{1}{u\kappa'(u)}\geq\left(\frac{1}{q-1}+1-\frac{q'}{p'}\right)^{-1}\frac{1}{\kappa(u)}=\frac{p}{q'\kappa(u)},$$

so $\Theta''(s)\geq p/(q's)$ for sufficiently large $s$. For such $s$, by the mean-value theorem and the fact that $\Theta'''<0$ (will show this at the end of the proof), we get

$$\overline{G}(s)\geq\Theta''(s+\gamma(s))\cdot\frac{\gamma(s)^2}{2}-\frac{\gamma(0)}{2}\geq\frac{p\gamma(s)^2}{q'(s+\gamma(s))}-\frac{\gamma(0)}{2}.$$

But by (4.4) and the first inequality in (4.12),

$$\gamma(s)=\frac{1+\gamma'(s)}{\Theta''(s)}\geq\frac{1}{\Theta''(s)}=-s\kappa'(s)\geq\left(1-\frac{q'}{p'}\right)s$$

for all $s$. Plugging this into the previous inequality, we see that $\overline{G}(s)$, and hence also $G(s)$, is positive for sufficiently large $s$. So, suppose that $G(s)<0$ for some (intermediate) $s>0$; therefore, there exists $s_0>0$ at which $\overline{G}$ attains its minimum. But

$$\begin{aligned}\overline{G}(s_0)&=\int_{s_0}^{s_0+\gamma(s_0)}\Theta'(s)\mathrm{d}s-\Theta'(s_0)\gamma(s_0)-\frac{\gamma(0)}{2}\\&\geq\frac{\Theta'(s_0)+\Theta'(s_0+\gamma(s_0))}{2}\cdot\gamma(s_0)-\Theta'(s_0)\gamma(s_0)-\frac{\gamma(0)}{2}\\&=\frac{\Theta'(s_0+\gamma(s_0))-\Theta'(s_0)}{2}\cdot\gamma(s_0)-\frac{\gamma(0)}{2},\end{aligned}$$

where in the second passage we have used the inequality $\Theta''' < 0$. However, the condition $\overline{G}'(s_0) = 0$ implies

$$(1+\gamma'(s_0))(\Theta'(s_0+\gamma(s_0)) - \Theta'(s_0)) - \Theta''(s_0)\gamma(s_0) = 0,$$

which, by (4.4) and the inequality $1+\gamma'(s_0) > 0$ is equivalent to $\Theta'(s_0+\gamma(s_0)) - \Theta'(s_0) = 1$. Plugging this above, we get $\overline{G}(s_0) = (\gamma(s_0) - \gamma(0))/2$, which is positive, since $\gamma$ is increasing. This is a contradiction, which shows that $G$ is positive on the whole $[0,\infty)$.

It remains to show that $\Theta''' < 0$, but this is due to the following two facts. First, by a direct differentiation we have $\Theta''(\kappa(u))(-u\kappa'(u)) = 1$; second, the function $u \mapsto -u\kappa'(u)$ is decreasing on $(0, 1/2]$ (see the proof of Lemma 4.3). □

We are ready for the proof of the martingale inequalities.

*Proof of Theorem 4.2.* Fix $t \geq 0$ and an arbitrary pair $(X, Y)$ as in the statement. By $1^\circ$, $2^\circ$ and Lemma 2.1, there is a sequence $(\tau_n)_{n\geq 1}$ of stopping times increasing to infinity such that $\mathbb{E}V(X_{\tau_n\wedge t}, Y_{\tau_n\wedge t}) \leq 0$ for each $n$. So, arguing as in Section 2, if $A$ is an arbitrary event of positive probability, then

$$\int_A |Y_{\tau_n\wedge t}|\mathrm{d}\mathbb{P} \leq \mathbb{E}\Theta(|X_{\tau_n\wedge t}|) + \frac{\gamma(0)\mathbb{P}(A)}{2}.$$

Letting $n \to \infty$, we obtain

$$\int_A |Y_t|\mathrm{d}\mathbb{P} \leq \mathbb{E}\Theta(|X_t|) + \frac{\gamma(0)\mathbb{P}(A)}{2}. \tag{4.13}$$

Next we apply (4.1) with $u = X_t^*(s)$ and $v = (\lambda s)^{q/p-1}$, where $\lambda$ is an auxiliary positive parameter. We obtain

$$\begin{aligned}
\mathbb{E}\Theta(|X_t|) &= \int_0^1 \Theta(X_t^*(s))\mathrm{d}s \\
&\leq \frac{\lambda^{q/p-1}}{q}\int_0^1 (s^{1/p}X_t^*(s))^q\frac{\mathrm{d}s}{s} + \int_0^\infty \Theta\left(\left(-(\lambda s)^{1-q/p}\ln(2\lambda s)\right)_+^{1/(q-1)}\right) \\
&\qquad - \frac{1}{q}\int_0^\infty \left(-(\lambda s)^{1-q/p}\ln(2\lambda s)\right)_+^{1/(q-1)}(-\ln(2\lambda s))\mathrm{d}s \\
&= \frac{\lambda^{q/p-1}}{q}\|X_t\|_{p,q}^q + \lambda^{-1}\int_0^\infty \Theta\left(\kappa(s)\right)\mathrm{d}s - \frac{\lambda^{-1}}{q}\int_0^\infty \kappa(s)(-\ln(2s))\mathrm{d}s.
\end{aligned}$$

Integrating by parts and using the definition of $\Theta$, we get

$$\begin{aligned}
\int_0^\infty \Theta(\kappa(s))\mathrm{d}s &= -\int_0^\infty s\Theta'(\kappa(s))\kappa'(s)\mathrm{d}s \\
&= \int_0^\infty s\ln(2s)\kappa'(s)\mathrm{d}s = -\int_0^\infty \kappa(s)(1+\ln(2s))\mathrm{d}s.
\end{aligned}$$

But

$$\int_0^\infty \kappa(s)\mathrm{d}s = \int_0^\infty t^{1/(q-1)}(e^{-t}/2)^{q'/p'}\mathrm{d}t = 2^{-q'/p'}\left(\frac{p'}{q'}\right)^{q'}\Gamma(q')$$

and, similarly,

$$\int_0^\infty \kappa(s)(-\ln(2s))\mathrm{d}s = p'\cdot 2^{-q'/p'}\left(\frac{p'}{q'}\right)^{q'}\Gamma(q'),$$

so plugging this above, we get

$$\mathbb{E}\Theta(|X_t|) \leq \frac{\lambda^{q/p-1}}{q}\|X\|_{p,q}^q + \lambda^{-1}\cdot 2^{-q'/p'}\left(\frac{p'}{q'}\right)^{q'}\Gamma(q')\cdot\frac{q-p}{q(p-1)}.$$

We minimize the right-hand side over $\lambda$: the choice $\lambda = \|X\|_{p,q}^{-p}C_{p,q}^{p/(q-1)}$ yields

$$\mathbb{E}\Theta(|X_t|) \leq C_{p,q}^{(q-p)/(q-1)}\cdot\frac{\|X_t\|_{p,q}^p}{p}.$$

Combining this with (4.13) and recalling that $\gamma(0)/2 = C_{p,q}^{q'}/p'$, we get

$$\int_A |Y_t|\mathrm{d}\mathbb{P} \leq C_{p,q}^{(q-p)/(q-1)}\cdot\frac{\|X_t\|_{p,q}^p}{p} + C_{p,q}^{q'}\cdot\frac{\mathbb{P}(A)}{p'}.$$

Now we exploit an additional homogenization argument: we apply the above estimate to the pair $(X/\mu, Y/\mu)$, where $\mu = C_{p,q}^{1/(q-1)}\|X\|_{p,q}^{-1}\mathbb{P}(A)^{1/p}$. The obtained inequality becomes

$$\int_A |Y_t|\mathrm{d}\mathbb{P} \leq C_{p,q}\|X\|_{p,q}\mathbb{P}(A)^{1-1/p},$$

so the claim follows, since $A$ was arbitrary. The sharpness has already been shown above. □

*Remark* 4.6. Let us briefly sketch some steps which have led to the special function $\Theta$. The indication is hidden in the estimates (2.10) and (2.11). Namely, we have identified the almost extremal pairs $(X, Y)$, or rather $(f, g)$ in (1.6) specialized to the context of martingale transforms/stochastic integrals. In particular, the non-increasing rearrangement of the extremal $X_t$ is close to $\kappa$. For the extremal pairs, we must have (almost) equality in (2.10), and both sides of (2.11) should also be (almost) equal for $u = X_t^*(s)$ and $v = s^{q/p-1}$ (for all $s > 0$). The latter occurs if $\Theta'(u) = u^{q-1}v$, which leads to the equation $\Theta'(X_t^*(s)) = (X_t^*(s))^{q-1}s^{q/p-1}$. Plugging $X_t^* = \kappa$ gives $\Theta'(\kappa(s)) = -\ln(2s)$, which yields the special function $\Theta$ considered above.

## 5. An application: weak-type inequalities for Fourier multipliers

Inequalities for differentially subordinate martingales lead to the corresponding results for the class of the so-called Lévy multipliers, introduced in [3, 4]. Furthermore, such estimates are optimal for the real part of the Beurling–Ahlfors operator, a fundamental object for the study of quasiconformal mappings and geometric function theory (cf. [2]). This interplay between the probabilistic and analytic contexts is well-known, so we will be brief. For clarity, it is convenient to split the material into a few separate subsections.

5.1. **Fourier multipliers and their stochastic representation.** For a given bounded function $m : \mathbb{R}^d \to \mathbb{C}$, there exists a uniquely bounded linear operator $T_m$ on $L^2(\mathbb{R}^d)$, called the Fourier multiplier with the symbol $m$, which is defined, in terms of Fourier transforms, by the identity $\widehat{T_m f} = m\widehat{f}$. By Plancherel's theorem, the norm of $T_m$ on $L^2(\mathbb{R}^d)$ is equal to $\|m\|_\infty$, and there is a natural problem to investigate those symbols $m$, for which the associated multipliers extend to bounded operators on (all or some) $L^p(\mathbb{R}^d)$, or some other function spaces. This problem seems to be too hard to study in such a generality; typically, one restricts oneself to a fixed class of symbols, enjoying some size/regularity conditions. We will study the so-called Lévy multipliers, which can be defined as follows. Let $\nu$ be a Lévy measure on $\mathbb{R}^d$, i.e., a nonnegative Borel measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and the integrability requirement

$$\int_{\mathbb{R}^d} \min\{|x|^2, 1\}\nu(\mathrm{d}x) < \infty.$$

Assume further that $\mu$ is a finite Borel measure on the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$ and fix two Borel functions $\phi$ on $\mathbb{R}^d$ and $\psi$ on $\mathbb{S}$ with values in $\mathbb{R}$. We define $m = m_{\phi,\psi,\mu,\nu}$ on $\mathbb{R}^d$ by

$$m(\xi) = \frac{\frac{1}{2}\int_{\mathbb{S}} \langle \xi, \theta\rangle^2 \psi(\theta)\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\phi(x)\nu(\mathrm{d}x)}{\frac{1}{2}\int_{\mathbb{S}} \langle \xi, \theta\rangle^2 \mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\nu(\mathrm{d}x)} \tag{5.1}$$

if the denominator is not 0, and $m(\xi) = 0$ otherwise; $\langle \cdot, \cdot\rangle$ is the scalar product in $\mathbb{R}^d$.

This class contains many important examples; we present here two multipliers and refer the reader to [4] for the extended list. Let $e_1$, $e_2$, ..., $e_d$ be the collection of unit vectors in $\mathbb{R}^d$ and let $\delta_{e_j}$ be the Dirac measure concentrated on $e_j$. If we take $\nu = 0$, $\mu = \delta_{e_1} + \delta_{e_2} + \dots \delta_{e_d}$ and consider $\psi$ which is equal to 1 on $e_i$, $-1$ on $e_j$ and vanishes for all other $e_k$'s, then $m(\xi) = (\xi_i^2 - \xi_j^2)/|\xi|^2$, i.e., $T_m$ is the combination $R_i^2 - R_j^2$ of second-order Riesz transforms on $\mathbb{R}^d$ (cf. [27]). In particular, setting $d = 2$, $i = 1$ and $j = 2$, we obtain the operator $R_1^2 - R_2^2$, the real part of the so-called Beurling–Ahlfors operator $\mathcal{B} = R_1^2 - R_2^2 + 2iR_1R_2$ on the plane.

We will prove the following statement.

**Theorem 5.1.** *For any Fourier multiplier $T_m$ associated with the symbol $m$ of the form* (5.1) *we have*

$$\|T_m f\|_{L^{p,q}(\mathbb{R}^d)} \leq C_{p',q'}\|f\|^{1/p}_{L^1(\mathbb{R}^d)}\|f\|^{1/p'}_{L^\infty(\mathbb{R}^d)}, \qquad 2 \leq q \leq p < \infty, \tag{5.2}$$

*and*

$$\|T_m f\|_{L^{p,\infty}(\mathbb{R}^d)} \leq C_{p,q}\|f\|_{L^{p,q}(\mathbb{R}^d)}, \qquad 1 < p < q \leq 2. \tag{5.3}$$

*In both estimates the constants are the best possible: for the real part of the Beurling-Ahlfors operator they cannot be replaced by smaller numbers.*

This result will be obtained by probabilistic methods: there is a beautiful connection between the class (5.1) and differentially subordinate martingales. Let $m$ be the multiplier as in (5.1), with the corresponding parameters $\phi$, $\psi$, $\mu$ and $\nu$. Assume in addition that $\nu(\mathbb{R}^d)$ is finite and nonzero. Then for any $s < 0$ there is a Lévy process $(X_{s,t})_{t\in[s,0]}$ with $X_{s,s} \equiv 0$, for which Lemmas 5.2 and 5.3 below hold true. To state these, we need some notation. For a given $f \in L^\infty(\mathbb{R}^d)$, define the corresponding parabolic extension $\mathcal{U}_f$ to $(-\infty, 0] \times \mathbb{R}^d$ by

$$\mathcal{U}_f(s, x) = \mathbb{E}f(x + X_{s,0}).$$

Next, fix $x \in \mathbb{R}^d$, $s < 0$ and let $f, \phi \in L^\infty(\mathbb{R}^d)$. We introduce the processes $F = (F_t^{x,s,f})_{s\leq t\leq 0}$ and $G = (G_t^{x,s,f,\phi})_{s\leq t\leq 0}$ by

$$\begin{aligned} F_t &= \mathcal{U}_f(t, x + X_{s,t}), \\ G_t &= \sum_{s<u\leq t} \big[(F_u - F_{u-}) \cdot \phi(X_{s,u} - X_{s,u-})\big] \\ &\quad - \int_s^t \int_{\mathbb{R}^d} \big[\mathcal{U}_f(v, x + X_{s,v-} + z) - \mathcal{U}_f(v, x + X_{s,v-})\big]\phi(z)\nu(\mathrm{d}z)\mathrm{d}v. \end{aligned} \tag{5.4}$$

Now, fix $s<0$ and define the operator $\mathcal{S}=\mathcal{S}^{s,\phi,\nu}$ by the bilinear form

$$\int_{\mathbb{R}^d} \mathcal{S}f(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\big[G_0^{x,s,f,\phi} g(x+X_{s,0})\big]\mathrm{d}x, \tag{5.5}$$

where $f,\, g\in C_0^\infty(\mathbb{R}^d)$. We have the following facts, proved in [3] and [4].

**Lemma 5.2.** *For any fixed $x$, $s$, $f$, $\phi$ as above, the processes $F^{x,s,f}$, $G^{x,s,f,\phi}$ are martingales with respect to $(\mathcal{F}_t)_{s\leq t\leq 0}=(\sigma(X_{s,t}:s\leq t))_{s\leq t\leq 0}$. Furthermore, if $\|\phi\|_\infty\leq 1$, then $G^{x,s,f,\phi}$ is differentially subordinate to $F^{x,s,f}$.*

Let us stress here that $\phi$, and hence also $G$, are complex valued. The aforementioned representation of Fourier multipliers in terms of Lévy processes is as follows.

**Lemma 5.3.** *Let $1<p<\infty$ and $d\geq 2$. The operator $\mathcal{S}^{s,\phi,\nu}$ is well defined and extends to a bounded operator on $L^p(\mathbb{R}^d)$, which can be expressed as the Fourier multiplier with the symbol*

$$\begin{aligned} M(\xi) &= M_{s,\phi,\nu}(\xi)\\ &= \left[1-\exp\left(2s\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz)\right)\right]\frac{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\phi(z)\nu(dz)}{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz)}\end{aligned}$$

*if $\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz)\neq 0$, and $M(\xi)=0$ otherwise.*

5.2. **Proof of** (5.2)**.** We may and do assume that at least one of the measures $\mu$, $\nu$ is nonzero. We split the reasoning into three parts.

*Step 1.* We will first establish an auxiliary $\Phi$ estimate for the multipliers with the symbols of the special form

$$M_{\phi,\nu}(\xi)=\frac{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\phi(z)\nu(\mathrm{d}z)}{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(\mathrm{d}z)}. \tag{5.6}$$

Assume that $0<\nu(\mathbb{R}^d)<\infty$; then the above representation in terms of Lévy processes is applicable. Fix $s<0$ and complex-valued functions $f,\, g\in C_0^\infty(\mathbb{R}^d)$. By homogeneity, we may and do assume that $\|f\|_{L^\infty(\mathbb{R}^d)}=1$. Let $\tilde{\Phi}$ be the Legendre transform of $\Phi$: $\tilde{\Phi}(s)=\sup_{t\geq 0}(st-\Phi(t))$. Then, by Young's inequality, (2.8) and Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}\big[G_0^{x,s,f,\phi}g(x+X_{s,0})\big]\mathrm{d}x &\leq \int_{\mathbb{R}^d}\mathbb{E}\Phi(|G_0^{x,s,f,\phi}|)\mathrm{d}x+\int_{\mathbb{R}^d}\mathbb{E}\tilde{\Phi}(|g(x+X_{s,0})|)\mathrm{d}x\\ &\leq c_{p,q}\int_{\mathbb{R}^d}\mathbb{E}|F_0^{x,s,f,\phi}|\mathrm{d}x+\mathbb{E}\int_{\mathbb{R}^d}\tilde{\Phi}(|g(x+X_{s,0})|)\mathrm{d}x\\ &= c_{p,q}\int_{\mathbb{R}^d}|f(x)|\mathrm{d}x+\int_{\mathbb{R}^d}\tilde{\Phi}(|g(x)|)\mathrm{d}x.\end{aligned}$$

By (5.5), this gives

$$\int_{\mathbb{R}^d}\Big[\mathcal{S}f(x)g(x)-\tilde{\Phi}(|g(x)|)\Big]\mathrm{d}x\leq c_{p,q}\|f\|_{L^1(\mathbb{R}^d)}.$$

By the definition of the Legendre transform, one can take $g$ so that $\mathcal{S}f(x)g(x)-\tilde{\Phi}(|g(x)|)=\Phi(|\mathcal{S}f(x)|)$. But such a function does not have to belong to $C_0^\infty(\mathbb{R}^d)$; nevertheless, for any $\varepsilon>0$ we may pick $g$ so that

$$\int_{\mathbb{R}^d}\Big[\mathcal{S}f(x)g(x)-\tilde{\Phi}(|g(x)|)\Big]\mathrm{d}x\geq(1-\varepsilon)\int_{\mathbb{R}^d}\Phi(|\mathcal{S}f(x)|)\mathrm{d}x.$$

Combining this with the previous estimate and letting $\varepsilon \to 0$, we obtain

$$\int_{\mathbb{R}^d} \Phi(|\mathcal{S}f(x)|)\mathrm{d}x \leq c_{p,q}\|f\|_{L^1(\mathbb{R}^d)}. \tag{5.7}$$

Now if we let $s \to -\infty$, then $M_{s,\phi,\nu}$ converges pointwise to the multiplier $M_{\phi,\nu}$ given by (5.6). By Plancherel's theorem, $\mathcal{S}^{s,\phi,\nu}f \to T_{M_{\phi,\nu}}f$ in $L^2(\mathbb{R}^d)$ and hence there is a sequence $(s_n)_{n=1}^\infty$ converging to $-\infty$ such that $\lim_{n\to\infty}\mathcal{S}^{s_n,\phi,\nu}f \to T_{M_{\phi,\nu}}f$ almost everywhere. Thus Fatou's lemma combined with (5.7) yields

$$\int_{\mathbb{R}^d} \Phi(|T_{M_{\phi,\nu}}f(x)|)\mathrm{d}x \leq c_{p,q}\|f\|_{L^1(\mathbb{R}^d)}. \tag{5.8}$$

*Step 2.* Now we turn to the general multipliers as in (5.1) and drop the assumption $0 < \nu(\mathbb{R}^d) < \infty$. For a given $\varepsilon > 0$, define a Lévy measure $\nu_\varepsilon$ in polar coordinates $(r,\theta) \in (0,\infty) \times \mathbb{S}$ by

$$\nu_\varepsilon(\mathrm{d}r\mathrm{d}\theta) = \varepsilon^{-2}\delta_\varepsilon(\mathrm{d}r)\mu(d\theta).$$

Here $\delta_\varepsilon$ denotes Dirac measure on $\{\varepsilon\}$. Next, consider a multiplier $M_{\varepsilon,\phi,\psi,\mu,\nu}$ as in (5.6), in which the Lévy measure is $1_{\{|x|>\varepsilon\}}\nu+\nu_\varepsilon$ and the jump modulator is given by $1_{\{|x|>\varepsilon\}}\phi(x)+1_{\{|x|=\varepsilon\}}\psi(x/|x|)$. Note that this Lévy measure is finite and nonzero, at least for sufficiently small $\varepsilon$. If we let $\varepsilon \to 0$, we see that

$$\begin{aligned}\int_{\mathbb{R}^d}[1-\cos\langle\xi,x\rangle]\psi(x/|x|)\nu_\varepsilon(\mathrm{d}x) &= \int_{\mathbb{S}}\langle\xi,\theta\rangle^2\phi(\theta)\frac{1-\cos\langle\xi,\varepsilon\theta\rangle}{\langle\xi,\varepsilon\theta\rangle^2}\mu(d\theta)\\ &\to \frac{1}{2}\int_{\mathbb{S}}\langle\xi,\theta\rangle^2\phi(\theta)\mu(\mathrm{d}\theta)\end{aligned}$$

and, consequently, $M_{\varepsilon,\phi,\psi,\mu,\nu} \to m_{\phi,\psi,\mu,\nu}$ pointwise. So, by Fatou's lemma, (5.8) yields

$$\int_{\mathbb{R}^d} \Phi(|T_{m_{\phi,\psi,\mu,\nu}}f(x)|)\mathrm{d}x \leq c_{p,q}\|f\|_{L^1(\mathbb{R}^d)}.$$

*Step 3.* We complete the proof. By (2.6) and the above estimate, setting $m = m_{\phi,\psi,\mu,\nu}$,

$$\begin{aligned}\int_0^\infty ((T_mf)^*(s))^q s^{q/p-1}\mathrm{d}s &= \|f\|_{L^1(\mathbb{R}^d)}^{q/p-1}\int_0^\infty ((T_mf)^*(s))^q\left(s/\|f\|_{L^1(\mathbb{R}^d)}\right)^{q/p-1}\mathrm{d}s\\ &\leq \|f\|_{L^1(\mathbb{R}^d)}^{q/p-1}\left[\int_0^\infty \Phi\big((T_mf)^*(s)\big)\mathrm{d}s + \int_0^\infty \Psi\big(s/\|f\|_{L^1(\mathbb{R}^d)}\big)\mathrm{d}s\right]\\ &= \|f\|_{L^1(\mathbb{R}^d)}^{q/p-1}\left[\int_{\mathbb{R}^d}\Phi(|T_mf|)\mathrm{d}x + \|f\|_{L^1(\mathbb{R}^d)}\int_0^\infty \Psi(s)\mathrm{d}s\right]\\ &\leq \|f\|_{L^1(\mathbb{R}^d)}^{q/p}\left[c_{p,q} + \int_0^{1/2}\Psi(s)\mathrm{d}s\right] = C_{p',q'}^q\|f\|_{L^1(\mathbb{R}^d)}^{q/p}.\end{aligned}$$

This is the desired claim.

5.3. **Proof of** (5.3)**.** We deduce the estimate from (5.2) by a duality argument, analogous to that used above in the context of stochastic integrals. Fix a function $f$ on $\mathbb{R}^d$ and let $A$ be a subset of $\mathbb{R}^d$ of positive measure. We have

$$\int_A |T_mf(x)|\mathrm{d}x = \int_{\mathbb{R}^d} T_mf(x)\overline{g(x)}\mathrm{d}x,$$

where $g = \chi_A T_m f/|T_m f|$ (we use the convention $0/0 = 0$). By Plancherel's theorem,

$$\int_A |T_m f(x)|\mathrm{d}x = \int_{\mathbb{R}^d} \widehat{T_m f}(\xi)\overline{\widehat{g}(\xi)}\mathrm{d}\xi = \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{T_{\bar{m}} g}(\xi)}\mathrm{d}\xi,$$

where $\bar{m}$ is the symbol obtained by replacing $\psi$ and $\phi$ with the complex conjugates $\bar{\psi}$ and $\bar{\phi}$; in particular, $\bar{m}$ is also of the form (5.1). But $g$ is bounded by 1; therefore, applying Hardy–Littlewood–Polyá inequality, Hölder's inequality and (5.2), we get

$$\begin{aligned}\int_A |T_m f(x)|\mathrm{d}x &\leq \int_0^\infty f^*(s)(T_{\bar{m}}g)^*(s)\mathrm{d}s \\ &\leq \left(\int_0^\infty (s^{1/p} f^*(s))^q \frac{\mathrm{d}s}{s}\right)^{1/q} \left(\int_0^\infty (s^{1/p'}(T_{\bar{m}}g)^*(s))^{q'} \frac{\mathrm{d}s}{s}\right)^{1/q'} \\ &\leq C_{p,q}\|g\|_{L^1(\mathbb{R}^d)}^{1/p'}\|f\|_{L^{p,q}(\mathbb{R}^d)}.\end{aligned}$$

However, we have $\|g\|_{L^1(\mathbb{R}^d)} \leq |A|$; this yields (5.3), since $A$ was arbitrary.

5.4. **Sharpness.** Now we will prove that the constants in the estimates (5.2) and (5.3) are optimal for any dimension $d \geq 2$. It is enough to focus on the sharpness of the weaker bound (5.3) (indeed: if the constant in (5.2) could be improved for some $p$, $q$, then (5.3) would not be sharp for $p'$, $q'$). This will be done with the use of laminates, important family of probability measures on matrices. It is convenient to split the reasoning into several separate parts. For the sake of convenience and to make the presentation as self contained as possible, we recall the preliminaries on laminates and their connections to martingales from [8].

*Laminates.* Assume that $\mathbb{R}^{m\times n}$ stands for the space of all real matrices of dimension $m \times n$ and $\mathbb{R}^{n\times n}_{sym}$ denote the subclass of $\mathbb{R}^{n\times n}$ which consists of all symmetric matrices of dimension $n \times n$.

**Definition 5.4.** A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called *rank-one convex*, if for all $A, B \in \mathbb{R}^{m\times n}$ with rank $B = 1$, the function $t \mapsto f(A + tB)$ is convex.

Let $\mathcal{P} = \mathcal{P}(\mathbb{R}^{m\times n})$ be the class of all compactly supported probability measures on $\mathbb{R}^{m\times n}$. For a measure $\nu \in \mathcal{P}$, we define its center of mass by

$$\overline{\nu} = \int_{\mathbb{R}^{m\times n}} X d\nu(X).$$

**Definition 5.5.** We say that a measure $\nu \in \mathcal{P}$ is a *laminate* and write $\nu \in \mathcal{L}$, if

$$f(\overline{\nu}) \leq \int_{\mathbb{R}^{m\times n}} f\mathrm{d}\nu$$

for all rank-one convex functions $f$. The set of laminates with center 0 is denoted by $\mathcal{L}_0(\mathbb{R}^{m\times n})$.

The key observation is that laminates can be regarded as probability measures that record the distribution of the gradients of smooth maps: see Corollary 5.9 below. We briefly explain this and refer the reader to the works [15], [21] and [30] for full details.

**Definition 5.6.** Let $U$ be a subset of $\mathbb{R}^{2\times 2}$ and let $\mathcal{PL}(U)$ denote the smallest class of probability measures on $U$ which

(i) contains all measures of the form $\lambda\delta_A + (1 - \lambda)\delta_B$ with $\lambda \in [0, 1]$ and satisfying $\mathrm{rank}(A - B) = 1$;

(ii) is closed under splitting in the following sense: if $\lambda\delta_A + (1-\lambda)\nu$ belongs to $\mathcal{PL}(U)$ for some $\nu \in \mathcal{P}(\mathbb{R}^{2\times 2})$ and $\mu$ also belongs to $\mathcal{PL}(U)$ with $\overline{\mu} = A$, then also $\lambda\mu + (1-\lambda)\nu$ belongs to $\mathcal{PL}(U)$.

The class $\mathcal{PL}(U)$ is called the *prelaminates* in $U$.

It follows immediately from the definition that the class $\mathcal{PL}(U)$ only contains atomic measures. Also, by a successive application of Jensen's inequality, we have the inclusion $\mathcal{PL} \subset \mathcal{L}$. The following are two well known lemmas in the theory of laminates; see [1], [15], [21], [30].

**Lemma 5.7.** *Let $\nu = \sum_{i=1}^N \lambda_i \delta_{A_i} \in \mathcal{PL}(\mathbb{R}^{2\times 2}_{sym})$ with $\overline{\nu} = 0$. Moreover, let $0 < r < \frac{1}{2}\min|A_i - A_j|$ and $\delta > 0$. For any bounded domain $\mathcal{B} \subset \mathbb{R}^2$ there exists $u \in W^{2,\infty}_0(\mathcal{B})$ such that $\|u\|_{C^1} < \delta$ and for all $i = 1, 2, \ldots, N$,*

$$\left|\{x \in \mathcal{B} : |D^2u(x) - A_i| < r\}\right| = \lambda_i|\mathcal{B}|.$$

**Lemma 5.8.** *Let $K \subset \mathbb{R}^{2\times 2}_{sym}$ be a compact convex set and suppose that $\nu \in \mathcal{L}(\mathbb{R}^{2\times 2}_{sym})$ satisfies $\operatorname{supp}\nu \subset K$. For any relatively open set $U \subset \mathbb{R}^{2\times 2}_{sym}$ with $K \subset U$, there exists a sequence $\nu_j \in \mathcal{PL}(U)$ of prelaminates with $\overline{\nu}_j = \overline{\nu}$ and $\nu_j \overset{*}{\rightharpoonup} \nu$, where $\overset{*}{\rightharpoonup}$ denotes weak convergence of measures.*

These two lemmas and a simple mollification argument yield the following statement, proved by Boros, Shékelyhidi Jr. and Volberg [8]. It exhibits the connection between laminates supported on symmetric matrices and second derivatives of functions. It will be our main tool in the proof of the sharpness. Let $\mathbb{D}$ denote the unit disc of $\mathbb{C}$.

**Corollary 5.9.** Let $\nu \in \mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})$. Then there exists a sequence $u_j \in C_0^\infty(\mathbb{D})$ with uniformly bounded second derivatives, such that

$$\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \varphi(D^2u_j(x))\,\mathrm{d}x \to \int_{\mathbb{R}^{2\times 2}_{sym}} \varphi\,\mathrm{d}\nu$$

for all continuous $\varphi : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}$.

*Biconvex functions and a special laminate.* The next step in our analysis is devoted to the introduction of a certain special laminate. We need some additional notation. A function $\zeta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is said to be *biconvex* if for any fixed $z \in \mathbb{R}$, the functions $x \mapsto \zeta(x,z)$ and $y \mapsto \zeta(z,y)$ are convex. Now, take the martingales $f$ and $g$ as in Remark 3.6 and consider the modified pair

$$(F_k, G_k) := \left(\frac{f_k + g_k}{2}, \frac{f_k - g_k}{2}\right), \qquad k = -1, 0, 1, \ldots, n.$$

This is a finite martingale starting from $(0,0)$, which has the following *zigzag* property: for any $k \geq 0$ we have $F_k = F_{k-1}$ with probability 1 or $G_k = G_{k-1}$ almost surely; that is, in each step $(F,G)$ moves either vertically, or horizontally. Indeed, this follows directly from the assumption that $g$ is a transform of $f$ by a predictable sequence with values in $\{-1,1\}$. This property combines nicely with biconvex functions: if $\zeta$ is such a function, then a successive application of Jensen's inequality gives

$$\mathbb{E}\zeta(F_n,G_n) \geq \mathbb{E}\zeta(F_{n-1},G_{n-1}) \geq \ldots \geq \mathbb{E}\zeta(F_{-1},G_{-1}) = \zeta(0,0). \tag{5.9}$$

The distribution of the terminal variable $(F_n, G_n)$ gives rise to a probability measure $\nu$ on $\mathbb{R}^{2\times 2}_{sym}$: put

$$\nu\left(\operatorname{diag}(x,y)\right) = \mathbb{P}\big((F_n,G_n)=(x,y)\big), \qquad (x,y)\in\mathbb{R}^2,$$

where $\operatorname{diag}(x,y)$ stands for the diagonal matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Observe that $\nu$ is a laminate of center 0. Indeed, if $\psi:\mathbb{R}^{2\times 2}\to\mathbb{R}$ is a rank-one convex, then $(x,y)\mapsto\psi(\operatorname{diag}(x,y))$ is biconvex and thus, by (5.9),

$$\int_{\mathbb{R}^{2\times 2}}\psi \mathrm{d}\nu = \mathbb{E}\psi(\operatorname{diag}(F_n,G_n)) \geq \psi(\operatorname{diag}(0,0)) = \psi(\bar{\nu}).$$

*Sharpness for Fourier multipliers, in the dimension $d=2$.* By Corollary 5.9, there is an appropriate functional sequence $u_j\in C_0^\infty(\mathbb{D})$. Taking the continuous function $\varphi:\mathbb{R}^{2\times 2}_{sym}\to\mathbb{R}$ given by $\varphi(A)=|A_{11}-A_{22}|$, we get

$$\frac{1}{|\mathbb{D}|}\int_{\mathbb{R}^2}\varphi(D^2u_j)\mathrm{d}x = \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}}\varphi(D^2u_j)\mathrm{d}x \xrightarrow{j\to\infty} \int_{\mathbb{R}^{2\times 2}_{sym}}\varphi\mathrm{d}\nu = \mathbb{E}|g_n|.$$

A similar argument shows that for $\varphi(A)=|A_{11}+A_{12}|$, we have

$$\liminf_{j\to\infty}\|\varphi(D^2u_j)\|_{L^{p,q}(\mathbb{D};\mu)} = \liminf_{j\to\infty}|\mathbb{D}|^{-1/p}\|\varphi(D^2u_j)\|_{L^{p,q}(\mathbb{C})} \leq \|f_n\|_{p,q},$$

where $\mu$ is the normalized Lebesgue measure on $\mathbb{D}$. Therefore, for sufficiently large $j$,

$$\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}}\left|\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}\right|\mathrm{d}x \geq (C_{p,q}-\varepsilon)|\mathbb{D}|^{-1/p}\|\Delta u_j\|_{L^{p,q}(\mathbb{C})}$$

and hence, setting $f=\Delta u_j$,

$$\|(R_1^2-R_2^2)f\|_{L^{p,\infty}(\mathbb{C})} \geq (C_{p,q}-\varepsilon)\|f\|_{L^{p,q}(\mathbb{C})}.$$

But $\varepsilon$ was arbitrary and $T_m=R_1^2-R_2^2$, the real part of the Beurling-Ahlfors operator, is a multiplier from class (5.1). This yields the desired sharpness.

*Sharpness for Fourier multipliers, dimension $d\geq 3$.* Suppose that for some $1<p<q\leq 2$ and $C>0$ we have

$$\int_A |(R_1^2-R_2^2)f(x)|\mathrm{d}x \leq C\|f\|_{L^{p,q}(\mathbb{R}^d)}|A|^{1/p'} \tag{5.10}$$

for all Borel subsets $A$ of $\mathbb{R}^d$ and all Borel functions $f:\mathbb{R}^d\to\mathbb{R}$. Note that the operator $R_1^2-R_2^2$ is a Fourier multiplier with a symbol belonging to (5.1). For $t>0$, define the dilation operator $\delta_t$ as follows: for any function $g:\mathbb{R}^2\times\mathbb{R}^{d-2}\to\mathbb{R}$, we let $\delta_t g(\xi,\eta)=g(\xi,t\eta)$; for any $A\subset\mathbb{R}^2\times\mathbb{R}^{d-2}$, let $\delta_t A=\{(\xi,t\eta):(\xi,\eta)\in A\}$. By (5.10), the operator $T_t:=\delta_t^{-1}\circ(R_1^2-R_2^2)\circ\delta_t$ satisfies

$$\begin{aligned}\int_A |T_tf(x)|\mathrm{d}x &= t^{d-2}\int_{\delta_t^{-1}A}|(R_1^2-R_2^2)\circ\delta_t f(x)|\mathrm{d}x \\ &\leq Ct^{d-2}||\delta_t f||_{L^{p,q}(\mathbb{R}^d)}|\delta_t^{-1}A|^{1/p'} = C||f||_{L^{p,q}(\mathbb{R}^d)}|A|^{1/p'}.\end{aligned} \tag{5.11}$$

Now fix $f\in L^p(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. It is not difficult to check that the Fourier transform $\mathcal{F}$ satisfies the identity $\mathcal{F}=t^{d-2}\delta_t\circ\mathcal{F}\circ\delta_t$ and hence the operator $T_t$ satisfies the identity

$$\widehat{T_tf}(\xi,\eta) = -\frac{\xi_1^2-\xi_2^2}{(|\xi|^2+t^2|\eta|^2)^{1/2}}\widehat{f}(\xi,\eta)$$

for $(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^{d-2}$. By Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to 0}\widehat{T_tf}(\xi,\eta)=\widehat{T_0f}(\xi,\eta)$$

in $L^2(\mathbb{R}^d)$, where $\widehat{T_0f}(\xi,\eta)=-\frac{\xi_1^2-\xi_2^2}{|\xi|^2}\widehat{f}(\xi,\eta)$. Combining this with Plancherel's theorem, we conclude that there is a sequence $(t_n)_{n\geq 1}$ decreasing to 0 such that $T_{t_n}f$ converges to $T_0f$ almost everywhere. Using Fatou's lemma and (5.11), we obtain

$$\int_A |T_0f(x)|\mathrm{d}x\leq C\|f\|_{L^{p,q}(\mathbb{R}^d)}|A|^{1/p'}. \tag{5.12}$$

Since $L^2(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, we easily verify that the above estimate holds true for all $f\in L^p(\mathbb{R}^d)$. Now pick an arbitrary function $\tilde{f}\in L^{p,q}(\mathbb{R}^2)$ and set $f(\xi,\zeta)=\tilde{f}(\xi)\chi_{[0,1]^{d-2}}(\zeta)$. We have $T_0f(\xi,\zeta)=(R_1^2-R_2^2)\tilde{f}(\xi)\chi_{[0,1]^{d-2}}(\zeta)$, because of the identity

$$\widehat{T_0f}(\xi,\eta)=-\frac{\xi_1^2-\xi_2^2}{|\xi|^2}\,\widehat{\tilde{f}}(\xi)\widehat{1_{[0,1]^{d-2}}}(\eta).$$

Plugging this into (5.12) and setting $A=B\times[0,1]^{d-1}$ for a fixed Borel subset $B$ of $\mathbb{R}^2$, we obtain

$$\int_B |(R_1^2-R_2^2)f|\mathrm{d}x\leq C\|\tilde{f}\|_{L^{p,q}(\mathbb{R}^2)}|B|^{1/p'}.$$

However, we have shown above that this implies $C\geq C_{p,q}$. The proof is complete.

## Acknowledgments

The author would like to thank an anonymous Referee for the careful reading of the paper and several helpful suggestions. The research was supported by Narodowe Centrum Nauki (Poland), grant 2018/30/Q/ST1/00072.

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