SHARP LORENTZ-NORM ESTIMATES FOR BMO MARTINGALES

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Abstract. Let X be a BMO martingale with continuous paths and let $2 \leq q \leq p < \infty$ be given parameters. The paper contains the proof of the Lorentz-norm inequality

$$\|X_\infty\|_{p,q} \leq 2^{-1/p}\,(p/q)^{(q+1)/q}\,\Gamma(q+1)^{1/q}\|X\|_{BMO},$$

and the constant is shown to be the best possible.

1. Introduction

Martingales of bounded mean oscillation play a prominent role in probability theory and stochastic analysis, and provide an efficient tool for the study of H_p spaces (e.g., via Fefferman's duality theorem [4], John-Nirenberg inequalities [8] or the integrability properties of associated exponential local martingales [9]). Although the origins of the BMO class go back to the analytic setting, the passage to the probabilistic context reveals some additional underlying structure and enables further applications (e.g., in financial mathematics: see [2], [3] or [5]). The purpose of this paper is to study a certain sharp Lorentz-norm estimate for this class of processes.

Let us fix the notation. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability zero. Let $X = (X_t)_{t\geq 0}$ be an adapted, continuous-path real-valued martingale. Following [6], the martingale X belongs to the space BMO, if it is uniformly integrable and its seminorm

$$\|X\|_{BMO} = \sup_{t\geq 0}\left\|\mathbb{E}\big[|X_\infty - X_t|^2 \,\big|\, \mathcal{F}_t\big]^{1/2}\right\|_\infty$$

is finite. The seminorm admits the equivalent formula

$$\|X\|^2_{BMO} = \sup_{t\geq 0}\operatorname{essup}\left(\mathbb{E}(X^2_\infty|\mathcal{F}_t) - X^2_t\right), \tag{1.1}$$

which makes the mean oscillation easier to handle (see below). The BMO martingales have very strong integrability properties (for an overview, see e.g. the book by Kazamaki [9]) and the question about the identification of best constants in the corresponding estimates has been studied intensively in the recent literature. Let us briefly discuss several important achievements in this direction: although most of them have been obtained in the analytic context, the passage to the probabilistic setting is immediate. One of the first results in this area is that of Slavin [15] and Slavin and Vasyunin [16], which gives the optimal constant in the integral form of John-Nirenberg inequality. Precisely, these

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works contain the proof of the exponential estimate

$$\mathbb{E}\exp(X_\infty - X_0) \leq \frac{\exp(-\|X\|_{BMO})}{1-\|X\|_{BMO}}.$$

Furthermore, this bound is sharp in the sense that for each $\varepsilon < 1$ there is a martingale X satisfying $\|X\|_{BMO} = \varepsilon$, $X_0 = 0$ and $\mathbb{E}\exp(X_\infty) = e^{-\varepsilon}/(1-\varepsilon)$. In particular, there is no exponential inequality of the above type when $\|X\|_{BMO} \geq 1$. The next important result, due to Vasyunin [20] and Vasyunin and Volberg [21], concerns the classical form the John-Nirenberg estimate. Namely, if $\varepsilon := \|X\|_{BMO} < \infty$, then we have

$$\mathbb{P}(|X_\infty - X_0| \geq \lambda) \leq \begin{cases} 1 & \text{if } 0 \leq \lambda \leq \varepsilon, \\ \varepsilon^2/\lambda^2 & \text{if } \varepsilon \leq \lambda \leq 2\varepsilon, \\ e^{2-\lambda/\varepsilon}/4 & \text{if } \lambda \geq 2\varepsilon, \end{cases}$$

and for each value of ε and λ, equality can be attained. Optimizing over λ, one obtains the sharp weak-type inequality

$$\|X_\infty - X_0\|_{p,\infty} \leq C_p \|X\|_{BMO}. \tag{1.2}$$

Here $C_p = 1$ if $0 < p < 2$ and $C_p = p2^{-2/p}e^{2/p-1}$ otherwise, and

$$\|\xi\|_{p,\infty} = \sup_{\lambda>0} \lambda \left[\mathbb{P}(|\xi| \geq \lambda)\right]^{1/p}$$

stands for the usual weak p-th quasinorm. We also mention here a related work of Slavin and Vasyunin [17], which establishes, among other things, the sharp L^p estimate

$$\|X_\infty - X_0\|_p \leq c_p \|X\|_{BMO}, \tag{1.3}$$

where $c_p = 1$ for $0 < p \leq 2$ and $c_p = \left(\frac{p}{2}\Gamma(p)\right)^{1/p}$ otherwise. We also refer the interested reader to the papers [7], [11], [12], [13] for the further development of the subject.

It should be emphasized here that all the results discussed have been obtained with the use of the so-called Bellman function method, a powerful technique used widely in various problems of analysis and probability. Roughly speaking, the approach reduces the problem of proving a given estimate for BMO martingales to the search of a certain special function, enjoying appropriate majorization and concavity conditions. We will briefly outline the method in Section 2 below, and we refer the interested reader to the works [7], [10], [14] and [18] for the general description.

The purpose of this paper is to continue the above direction of research. Motivated by (1.2) and (1.3), one may ask about best constants in the corresponding *Lorentz-norm* estimates for BMO martingales. We need some additional definitions. Given a random variable ξ, its nonincreasing rearrangement $\xi^* : (0,1] \to [0,\infty)$ is defined by

$$\xi^*(t) = \inf\left\{\lambda \geq 0 : \mathbb{P}(|\xi| > \lambda) \leq t\right\}.$$

Now, given $0 < p, q < \infty$, the Lorentz space $L^{p,q} = L^{p,q}(\Omega, \mathcal{F}, \mathbb{P})$ is the family of all (equivalence classes of) random variables ξ for which

$$\|\xi\|_{p,q} = \left(\int_0^1 (t^{1/p}\xi^*(t))^q \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty.$$

See e.g. [1] for more on the subject. The main result of this paper is the following.

Theorem 1.1. *Suppose that $2 \leq q \leq p < \infty$. Then for any continuous-path BMO martingale X we have*

$$\|X_\infty - X_0\|_{p,q} \leq 2^{-1/p} (p/q)^{(q+1)/q} \Gamma(q+1)^{1/q} \|X\|_{BMO}. \tag{1.4}$$

The constant is the best possible.

We would like to point out that the direct application of the Bellman function method is not possible for (1.4). One of the contributions of this paper is the introduction of an additional splitting argument which combined with the Bellman function method yields the claim. Unfortunately, our approach works in the limited range of exponents $2 \leq q \leq p < \infty$, the analysis for remaining p, q requires the development of new ideas.

The rest of the paper is organized as follows. In the next section we discuss briefly the Bellman function method and exhibit the obstacles which arise in the proof of (1.4). The Lorentz-norm estimate is established in Section 3. The last part of the paper is devoted to the sharpness of the inequality: we show that the constant cannot be replaced by a smaller one.

2. On the approach

As we have mentioned in the previous section, the Bellman function method allows to study inequalities for BMO martingales by constructing certain special functions. The argument goes as follows. Suppose that c is a constant and $V : \mathbb{R} \to [0, \infty)$ is a given Borel function. Assume further that we want to establish the estimate

$$\mathbb{E}V(X_\infty) \leq c \tag{2.1}$$

for all continuous-path BMO martingales X satisfying $\|X\|_{BMO} \leq 1$ and $X_0 = 0$. To handle this problem, we introduce, for any $\varepsilon > 0$, the parabolic domain

$$D_\varepsilon = \{(x, y) \in \mathbb{R}^2 \, : \, x^2 \leq y \leq x^2 + \varepsilon^2\}$$

and let $B : D_1 \to [0, \infty)$ be a continuous function enjoying the following requirements:

1° We have $B(0, y) \leq c$ for all $y \in [0, 1]$.

2° We have the majorization $B(x, x^2) \geq V(x)$ for all $x \in \mathbb{R}$.

3° The function B is locally concave, i.e., it is concave along any line segment entirely contained in D_1.

We have the following fact.

Theorem 2.1. *If there is a function B satisfying 1°, 2° and 3°, then* (2.1) *holds.*

Proof. Fix a BMO martingale X satisfying $\|X\|_{BMO} \leq 1$ and $X_0 = 0$. Suppose first that the BMO norm is less than 1: $\|X\|_{BMO} = \varepsilon < 1$, and consider the auxiliary process $Y_t = \mathbb{E}(X_\infty^2 | \mathcal{F}_t)$, $t \geq 0$. Note that the pair (X, Y) takes values in $D_\varepsilon \subset D_1$ and the composition $B(X, Y)$ makes sense: here (1.1) comes into play. The key argument is the application of Itô's formula to $B(X, Y)$; however, B does not have to be of class C^2, so an additional mollification argument is required (see e.g. [19], formula (5.3)). Namely, let g be a nonnegative C^∞ radial function on $\mathbb{R}^2$, supported on the unit ball and satisfying $\int_{\mathbb{R}^2} g = 1$. Given $\delta \in (0, (1-\varepsilon)/3)$, we consider the function $B^\delta : D_\varepsilon \to \mathbb{R}$ given by the convolution-type expression

$$B^\delta(x, y) = \int_{[-1,1]^2} B(x - \delta u, y + \delta - 2x\delta u + \delta^2 u^2 - \delta v) g(u, v) \mathrm{d}u\mathrm{d}v.$$

Note that the integrand is well-defined: we have

$$\big(y+\delta-2x\delta u+\delta^2u^2-\delta v\big)-(x-\delta u)^2=y-x^2+\delta-\delta v\in[0,1]$$

for $v\in[-1,1]$. The function B^δ is of class C^∞ and, by the very definition, it inherits the local concavity and nonnegativity. Furthermore, since g is radial,

$$\begin{aligned}
B^\delta(0,y)&=\int_{[-1,1]^2}B(-\delta u,y+\delta+\delta^2u^2-\delta v)g(u,v)\mathrm{d}u\mathrm{d}v\\
&=\int_{[-1,1]^2}\frac{B(-\delta u,y+\delta+\delta^2u^2-\delta v)+B(\delta u,y+\delta+\delta^2u^2-\delta v)}{2}g(u,v)\mathrm{d}u\mathrm{d}v\\
&\leq\int_{[-1,1]^2}B(0,y+\delta+\delta^2u^2-\delta v)g(u,v)\mathrm{d}u\mathrm{d}v\leq c.
\end{aligned}$$

Here in the third passage we have used the concavity of B along the horizontal line segment joining $(\pm\delta u,y+\delta+\delta^2u^2-\delta v)$, and the last passage is due to 1°. Consequently, Itô's formula gives, for any t,

$$B^\delta(X_t,Y_t)=B^\delta(X_0,Y_0)+I_1+I_2/2+I_3 \tag{2.2}$$

(note the essential inclusion $(X,Y)\in D_\varepsilon$), where

$$\begin{aligned}
I_1&=\int_0^t B^\delta_x(X_s,Y_{s-})\mathrm{d}X_s+\int_0^t B^\delta_y(X_s,Y_{s-})\mathrm{d}Y_s,\\
I_2&=\int_0^t B^\delta_{xx}(X_s,Y_{s-})\mathrm{d}\langle X,X\rangle_s+2\int_0^t B^\delta_{xy}(X_s,Y_{s-})\mathrm{d}\langle X,Y^c\rangle_s\\
&\quad+\int_0^t B^\delta_{yy}(X_s,Y_{s-})\mathrm{d}\langle Y^c,Y^c\rangle_s,\\
I_3&=\sum_{0<s\leq t}\Big[B^\delta(X_s,Y_s)-B^\delta(X_s,Y_{s-})-B^\delta_x(X_s,Y_{s-})\Delta Y_s\Big].
\end{aligned}$$

Here we have used the fact that X has continuous trajectories; however, on contrary, Y does not have to possess this regularity. This forces us to write the left limits Y_{s-}, the continuous part Y^c and the jump term I_3 in the expressions above.

Let us study the behavior of the terms I_1, I_2 and I_3. Both stochastic integrals in I_1 are local martingales; let $(\tau_n)_{n\geq1}$ be a common localizing sequence. Then, replacing t with $\tau_n\wedge t$ in I_1, we get $\mathbb{E}I_1=0$, by the properties of stochastic integrals. Furthermore, since B^δ is locally concave, the term I_2 (also, with t replaced by $\tau_n\wedge t$) is nonpositive: this can be seen by approximating the integrals with Riemann sums. Finally, by the concavity of the function $B^\delta(x,\cdot)$ for any fixed $x\in\mathbb{R}$, we conclude that each summand in I_3 (and hence also the whole I_3) is nonpositive. Thus, integrating both sides of (2.2), we get

$$\mathbb{E}B^\delta(X_{\tau_n\wedge t},Y_{\tau_n\wedge t})\leq\mathbb{E}B^\delta(X_0,Y_0)=\mathbb{E}B^\delta(0,Y_0)\leq c,$$

since B^δ satisfies 1° as well, as we have showed above. Therefore, letting $n\to\infty$, $t\to\infty$ and applying Fatou's lemma, we obtain $\mathbb{E}B^\delta(X_\infty,Y_\infty)\leq c$. However, B is continuous, so we have the pointwise convergence $B^\delta\to B$ as $\delta\to0$. Consequently, by Fatou's lemma again, the estimate $\mathbb{E}B(X_\infty,Y_\infty)\leq c$ holds. Now, we have obtained this bound under the assumption $\|X\|_{BMO}<1$ (see the beginning of the proof). However, if the BMO norm of X is equal to 1, then the estimate is still valid: by Fatou's lemma,

$$\mathbb{E}B(X_\infty,Y_\infty)\leq\liminf_{\varepsilon\uparrow1}\mathbb{E}B(\varepsilon X_\infty,\varepsilon^2Y_\infty)\leq c.$$

It remains to note that by the very definition of Y, we have $Y_\infty = X_\infty^2$ almost surely. Hence, by 2°, we obtain the desired estimate $\mathbb{E}V(X_\infty) \leq c$. □

The above interplay between the existence of the special function B and the validity of (2.1) is the key to the proofs of all the inequalities mentioned in the previous section. However, the Bellman function method has a serious limitation: it enables the study of estimates which are of the integral form (2.1) only. It is clear that because of the appearance of the Lorentz norm, the inequality (1.4) cannot be rewritten in such a shape. To overcome this difficulty, we will use a two-step procedure. First, suppose that Φ, $\Psi : [0,\infty) \to [0,\infty)$ are functions satisfying the Young-type estimate

$$u^q v^{q/p-1} \leq \Phi(u) + \Psi(v) \tag{2.3}$$

for all $u \geq 0$ and $v > 0$. Assuming that $X_0 = 0$ and $\|X\|_{BMO} < 1$ we get, by the direct integration,

$$\|X_\infty\|_{p,q}^q = \int_0^1 (X_\infty^*(s))^q s^{q/p-1} \mathrm{d}s \leq \int_0^1 \Phi\big(X_\infty^*(s)\big)\mathrm{d}s + \int_0^1 \Psi(s)\mathrm{d}s. \tag{2.4}$$

Note that the second integral $\int_0^1 \Psi(s)\mathrm{d}s$ is a constant. The second step is to prove the sharp bound of the form

$$\int_0^1 \Phi\big(X_\infty^*(s)\big)\mathrm{d}s \leq c_{p,q}. \tag{2.5}$$

Observe that this estimate is of the 'integral' form (2.1) and can be studied with the use of Bellman function method described above. Combining these two steps, we will get the desired claim. The argument may seem simple, but note that it is absolutely not clear whether the functions Φ, Ψ can be chosen so that we obtain the best constant in (1.4). We have managed to show that this is the case in the range $2 \leq q \leq p < \infty$. Actually, the choice will be highly nontrivial and the formulas for the functions will be quite complicated.

3. Proof of (1.4)

We start with the definition of two special Young functions (some steps which lead to their discovery are discussed in Remark 4.2 below). Fix $2 \leq q \leq p < \infty$ and introduce $\Phi = \Phi_{p,q}$, $\Psi = \Psi_{p,q} : [0,\infty) \to [0,\infty)$ by

$$\Phi(s) = 2^{1-q/p} \int_0^{s^q} \exp\left(\frac{p-q}{p} u^{1/q}\right) \mathrm{d}u$$

and

$$\Psi(t) = \begin{cases} (-\ln(2t))^q t^{q/p-1} - \Phi(\ln(2t)) & \text{if } t < 1/2, \\ 0 & \text{if } t \geq 1/2. \end{cases}$$

We will show that Φ and Ψ satisfy the appropriate version of Young's inequality. We include the straightforward proof for the sake of completeness.

Lemma 3.1. *The estimate* (2.3) *holds true.*

Proof. Fix $v > 0$, substitute $r = u^q$ and consider the function $\xi : [0,\infty) \to \mathbb{R}$, given by

$$\xi(r) = r v^{q/p-1} - \Phi(r^{1/q}) - \Psi(v).$$

We compute that $\xi'(r) = v^{q/p-1} - 2^{1-q/p}\exp\big((p-q)r^{1/q}/p\big)$. Therefore, if $v \geq 1/2$, then ξ is decreasing and $\xi(r) \leq \xi(0) = 0$. On the other hand, if $v < 1/2$, then ξ attains its maximal value at $r = (-\ln(2v))^q$: $\xi(r) \leq \xi((-\ln(2v))^q) = 0$. □

Now we will introduce the Bellman function associated with the estimate (2.5). This bound is of the form (2.1), with $V(x) = \Phi(|x|)$ and

$$c = c_{p,q} = 2^{-q/p}\,(p/q)^q\,\Gamma(q+1). \tag{3.1}$$

Recall the parabolic domain $D_1 = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 \leq y \leq x^2+1\}$ and let $B : D_1 \to \mathbb{R}$ be given by the formula

$$B(x,y) = \begin{cases} c_{p,q}y & \text{if } 2|x| \leq y \leq 2, \\ \Phi(\nu+|x|) - \nu\displaystyle\int_0^1 \Phi'\big(\nu+|x|-\ln u\big)\mathrm{d}u & \text{otherwise.}\end{cases}$$

Here, for brevity, we used the notation $\nu = \sqrt{1-y+x^2}-1$.

This function is continuous, it is even of class C^1: we have

$$\lim_{y\uparrow 2|x|} B(x,y) = |x|\int_0^1 \Phi'(-\ln u)\mathrm{d}u = c_{p,q}y$$

and

$$\lim_{y\uparrow 2|x|} B_y(x,y) = \frac{1}{2}\int_0^1 \Phi''(-\ln u)\mathrm{d}u = c_{p,q}.$$

Next, we will check that the function enjoys the required conditions 1°, 2° and 3°. The first property is obviously satisfied. The majorization 2° is also true: we have $B(x,x^2) = \Phi(|x|)$. The concavity condition 3° is verified in a separate lemma below.

Lemma 3.2. *The function B is locally concave on D_1.*

Proof. Since B is of class C^1, it is enough to check that D^2B, the Hessian matrix of B, is negative-definite (at all points from the interior of D_1, at which B is twice differentiable). If $2|x| < y < 2$, then there is nothing to prove: the Hessian is a zero matrix. If the point (x,y) satisfies $y < 2|x|$ or $y \geq 2$, then we easily see that there exists a (short) line segment of slope $a = 2x(1+\sqrt{1-y+x^2}/|x|)$ passing through (x,y) along which B is linear. Therefore,

$$B_{xx}(x,y) + 2aB_{xy}(x,y) + a^2B_{yy}(x,y) = 0. \tag{3.2}$$

Next, a direct differentiation and integration by parts yields

$$B_y(x,y) = \frac{1}{2}\int_0^1 \Phi''(\sqrt{1-y+x^2}+|x|-1-\ln u)\mathrm{d}u,$$

$$B_{xy}(x,y) = \frac{1}{2}\left(\frac{x}{\sqrt{1-y+x^2}} + \frac{x}{|x|}\right)\int_0^1 \Phi'''\Big(\sqrt{1-y+x^2}+|x|-1-\ln u\Big)\mathrm{d}u$$

and

$$B_{yy}(x,y) = -\frac{1}{4\sqrt{1-y+x^2}}\int_0^1 \Phi'''\Big(\sqrt{1-y+x^2}+x-1-\ln u\Big)\mathrm{d}u.$$

Hence $B_{xy}(x,y) = aB_{yy}(x,y)$, which combined with (3.2) yields $B_{xx}(x,y) = aB_{xy}(x,y)$ and implies that the determinant of $D^2B(x,y)$ is equal to 0. To show that the Hessian is negative definite, it remains to note that $B_{yy}(x,y) < 0$. This follows from the fact that

$$\Phi'''(s) = q2^{1-q/p}\exp\left(\frac{p-q}{p}s\right)s^{q-3}\left[\left(\frac{p-q}{p}\right)^2 s^2 + 2\left(\frac{p-q}{p}\right)(q-1) + (q-1)(q-2)\right]$$

is positive for all $s > 0$. Here the assumption $q \geq 2$ plays a key role. □

We are ready for the proof of the Lorentz-norm estimate.

Proof of (1.4). Pick an arbitrary continuous BMO martingale X. With no loss of generality, we may assume that $X_0 = 0$, replacing X with $X - X_0$ if necessary. Similarly, we may assume that $\|X\|_{BMO} = 1$, by homogeneity of (1.4). Applying (2.3), we obtain

$$\int_0^1 \big(X^*_\infty(t)\big)^q t^{q/p-1}\mathrm{d}t \leq \int_0^1 \Big[\Phi\big(X^*_\infty(t)\big) + \Psi(t)\Big]\mathrm{d}t = \mathbb{E}\Phi\big(|X_\infty|\big) + \int_0^1 \Psi(t)\mathrm{d}t.$$

The Bellman function method implies that the first integral does not exceed $c_{p,q}$ given by (3.1). Furthermore, we derive that

$$\begin{aligned}\int_0^1 \Psi(t)\mathrm{d}t &= \int_0^{1/2}(-\ln(2t))^q t^{q/p-1}\mathrm{d}t - \int_0^{1/2}\Phi(-\ln(2t))\mathrm{d}t\\ &= 2^{-q/p}\,(p/q)^{q+1}\,\Gamma(q+1) - c_{p,q},\end{aligned}$$

which combined with the above estimate gives

$$\int_0^1 \big(X^*_\infty(t)\big)^q t^{q/p-1}\mathrm{d}t \leq 2^{-q/p}\,(p/q)^{q+1}\,\Gamma(q+1).$$

This is precisely (1.4). □

4. Sharpness

Now we will show that the constant in (1.4) is the best possible: this will be accomplished by the construction of appropriate examples. We begin with an auxiliary notation. Let $W = (W_t)_{t\geq 0}$ be the standard one-dimensional Brownian motion started at zero and let $\overline{W}_t = \sup_{0\leq s\leq t}|W_s|$, $t \geq 0$, be the maximal function of W. Consider the stopping times

$$\sigma = \inf\{t > 0 : |W_t| = 1/2\}, \qquad \tau = \inf\big\{t > \sigma : |W_t| + 1 \leq \overline{W}_t \vee 1\big\},$$

with the standard convention $\inf\emptyset = +\infty$. Finally, put $X_t = W_{\tau\wedge t}$ for $t \geq 0$; then X is a continuous-path martingale, by Doob's optional sampling theorem. We will show that for this choice, both sides of (1.4) become equal.

Lemma 4.1. *The martingale X is uniformly integrable and satisfies $X_0 = 0$, $\|X\|_{BMO} = 1$ and $X^*_\infty(s) = (-\ln(2s))_+$ for $s \in (0,1]$.*

Proof. The equality $X_0 = 0$ is obvious. Now we will show that X is bounded in L^2. To this end, observe that the process

$$\zeta_t = \begin{cases} 0 & \text{if } t \leq \sigma,\\ (\overline{X}_t \vee 1)^2 - 2(\overline{X}_t \vee 1)|X_t| & \text{if } t > \sigma,\end{cases}$$

is a martingale; this follows at once from Itô's formula (note that after the time σ, the process X does not change its sign; thus the non-differentiability of $|\cdot|$ does not matter). Here, in analogy to the above notation, $\overline{X}$ is the maximal function of X. Consequently, by Doob's optional sampling theorem and the definition of τ, we obtain

$$\begin{aligned} 0 = -\mathbb{E}\zeta_{\sigma\vee t} &= \mathbb{E}\Big[2(\overline{X}_{\sigma\vee t}\vee 1)|X_{\sigma\vee t}| - (\overline{X}_{\sigma\vee t}\vee 1)^2\Big] \\ &\geq \mathbb{E}\Big[(\overline{X}_{\sigma\vee t}\vee 1)^2 - 2(\overline{X}_{\sigma\vee t}\vee 1)\Big] \geq \frac{1}{2}\mathbb{E}(\overline{X}_{\sigma\vee t}\vee 1)^2 - 2. \end{aligned}$$

Thus, by Lebesgue's monotone convergence theorem, we get

$$\mathbb{E}\overline{X}_\infty^2 \leq \mathbb{E}(\overline{X}_\infty \vee 1)^2 < \infty,$$

which yields the desired L^2-boundedness (and the uniform integrability) of X. In particular, the almost sure limit X_∞ exists. Next, apply Doob's optional sampling theorem to obtain $\mathbb{E}(\zeta_{\sigma\vee t}|\mathcal{F}_s) = \zeta_s \geq -X_s^2$ for arbitrary $s<t$. Letting $t\to\infty$ and using Lebesgue's dominated convergence theorem, the L^2 boundedness of X and the definition of τ, we get

$$\mathbb{E}(-X_\infty^2 + 1|\mathcal{F}_s) = \mathbb{E}(\zeta_\infty|\mathcal{F}_s) \geq -X_s^2.$$

This yields $\|X\|_{BMO}\leq 1$, since s was arbitrary. To get the formula for the nonincreasing rearrangement X_∞^*, consider the auxiliary martingale

$$\xi_t = \begin{cases} e & \text{if } t\leq\sigma, \\ 2\exp\left(\overline{X}_t\vee 1\right)\left(|X_t| + 1 - \overline{X}_t \vee 1\right) & \text{if } t>\sigma. \end{cases}$$

Fix $\lambda>0$ and introduce the stopping time $\eta = \inf\{t\geq 0 : \overline{X}_t > \lambda+1\}$. It follows directly from the definition of X that $\{|X_\infty|>\lambda\} = \{\eta<\infty\}$. Since the martingale $(\xi_{\eta\wedge t})_{t\geq 0}$ is bounded, we have $e = \mathbb{E}\xi_\eta$. Now on the set $\{\eta<\infty\}$ we have $\xi_\eta = 2e^{\lambda+1}$; on the other hand, on the complement $\{\eta=\infty\}$, we have $\xi_\eta = \xi_\infty = 2\exp\left(\overline{X}_\infty\vee 1\right)\left(|X_\infty|+1-\overline{X}_\infty\vee 1\right) = 0$, by the definition of τ. Thus, we finally obtain $e = 2e^{\lambda+1}\mathbb{P}(\eta<\infty)$, i.e.,

$$\mathbb{P}(|X_\infty|>\lambda) = e^{-\lambda}/2.$$

This yields the formula for X_∞^*. □

It remains to note that for the martingale X constructed above, both sides of (1.4) are equal. Indeed, we have

$$\begin{aligned} \|X_\infty\|_{p,q} = \left(\int_0^1 (-\ln(2s))_+^q s^{q/p-1}\mathrm{d}s\right)^{1/q} &= 2^{-1/p}\,(p/q)^{(q+1)/q}\,\Gamma(q+1)^{1/q} \\ &= 2^{-1/p}\,(p/q)^{(q+1)/q}\,\Gamma(q+1)^{1/q}\|X\|_{BMO}, \end{aligned}$$

which yields the desired sharpness.

Remark 4.2. There is a very natural question about the motivation for the formulas for the special Young functions Φ and Ψ used above. The first observation is that if such functions exist, then the intermediate bounds (2.4) and (2.5) should also be sharp. In other words, we postulate the existence of a martingale X for which equality is attained in both these estimates. Let us take a look at the first bound. If the equality holds there, then both sides of (2.3), with $u = X_\infty^*(s)$ and $v=s$, should also be equal. However, by a direct differentiation, the estimate (2.3) becomes an equality if $qu^{q-1}v^{q/p-1} = \Phi'(u)$, and coming back to the nonincreasing rearrangement X_∞^*, we obtain

$$qX_\infty^*(s)^{q-1}s^{q/p-1} = \Phi'(X_\infty^*(s)). \tag{4.1}$$

Therefore, if we knew the explicit formula for the nonincreasing rearrangement X_∞^*, this would lead us to the derivative of Φ and hence also the function Φ itself; the formula for the second function, Ψ, would then be immediate and would follow from the fact that equality holds in (2.3) for $u = X_\infty^*(s)$ and $v = s$.

So, all we need is the explicit formula for X_∞^*; here the second inequality (2.5) comes into play. After some thought and a little experimentation, one comes up with the exemplary martingale X presented above. Actually, the same martingale yields equality in (1.3) for $p \geq 2$. Plugging the identity $X_\infty^*(s) = (-\ln(2s))_+$ into (4.1), we obtain the functions Φ and Ψ defined at the beginning of Section 3.

We should point out here that the assumption $p \geq q \geq 2$ is essential here: without this condition, we still obtain the function Φ, but then the associated Bellman function does not have the required local concavity (see the end of proof of Lemma 3.2). In other words, roughly speaking, we either lose too much when splitting (1.4) into (2.4) and (2.5) (i.e., one should make the passage in one step), or the above exemplary X is not the right choice.

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