# SHARP LLOGL ESTIMATES FOR FOURIER MULITPLIERS

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Abstract. We study sharp local LlogL estimates for Fourier multipliers arising from modulation of jumps of Lévy processes. Precisely, we exhibit a large class of symbols $m$ on $\mathbb{R}^d$ such that the corresponding multiplier $T_m$ satisfies

$$\int_A |T_m f(x)| \mathrm{d}x \leq ||f||_{L \log L(A)},$$

for all Borel subsets $A$ of $\mathbb{R}^d$ and all Borel functions $f$ on $\mathbb{R}^d$ belonging to the appropriate class $L \log L$. In particular, this estimate holds true and is sharp for the real part of the Beurling-Ahlfors operator on $\mathbb{C}$. The proof rests on probabilistic methods.

## 1. Introduction

The motivation for the results obtained in this paper comes from a natural question about the action of Fourier multipliers on the class $L \log L$. Recall that for any bounded function $m : \mathbb{R}^d \to \mathbb{C}$, there is a unique bounded linear operator $T_m$ on $L_2(\mathbb{R}^d)$, called *the Fourier multiplier associated with the symbol* $m$, which is defined by the equality $\widehat{T_m f} = m\hat{f}$. A straightforward use of Plancherel's theorem shows that the norm of $T_m$ on $L_2(\mathbb{R}^d)$ is equal to $||m||_{L_\infty(\mathbb{R}^d)}$. A classical problem, which has been studied intensively in the literature, is to analyze those $m$, for which the associated multiplier extends to a bounded linear operator on $L_p(\mathbb{R}^d)$, $1 < p < \infty$. A related important question, on which we will focus in this work, is to compute the exact norm of a given multiplier, as an operator between given spaces. This problem is in general very difficult, and so far, it has been successfully treated only for a relatively small class of symbols. To the best of our knowledge, the first statement in this direction is that of Pichorides [28], which identifies the norm of the Hilbert transform $\mathcal{H}^{\mathbb{R}}$ as an operator on $L_p(\mathbb{R})$. Recall that the Hilbert transform on the line is given by the equality

$$\widehat{\mathcal{H}^{\mathbb{R}} f}(\xi) = -i \operatorname{sgn} \xi \cdot \widehat{f}(\xi), \qquad \xi \in \mathbb{R}.$$

Pichorides showed that if $1 < p < \infty$, then $||\mathcal{H}^{\mathbb{R}}||_{L_p(\mathbb{R}) \to L_p(\mathbb{R})} = \cot(\pi/2p^*)$, where $p^* = \max\{p, p/(p-1)\}$. This result was generalized to the higher dimensional setting by Iwaniec and Martin [17] and Bañuelos and Wang [6]. If $d \geq 1$ is a fixed integer, then the collection $R_1, R_2, \ldots, R_d$ of Riesz transforms (cf. Stein [29]) is defined by

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^d \setminus \{0\}, \, j = 1, 2, \ldots, d.$$

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These operators indeed generalize the Hilbert transform: if $d = 1$, then the family contains only one element, equal to $\mathcal{H}^{\mathbb{R}}$. The aforementioned result of Iwaniec, Martin, Bañuelos and Wang asserts that

$$||R_j||_{L_p(\mathbb{R}^d)\to L_p(\mathbb{R}^d)} = \cot\left(\frac{\pi}{2p^*}\right)$$

for all $d$, all $j \in \{1, 2, \ldots, d\}$ and all $1 < p < \infty$. For related sharp or almost sharp estimates for Hilbert and Riesz transforms, see e.g. Aarão and O'Neill [1], Davis [14], Janakiraman [18] and Osękowski [24], [27].

In the present paper we will be interested in a slightly different class of symbols, which can be obtained with the use of probabilistic methods (more precisely, by the modulation of jumps of certain Lévy processes). This class has been introduced and studied by Bañuelos and Bogdan [3] and Bañuelos, Bielaszewski and Bogdan [4]. Let $\nu$ be a Lévy measure on $\mathbb{R}^d$, i.e., a nonnegative Borel measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min\{|x|^2, 1\}\nu(\mathrm{d}x) < \infty.$$

Assume further that $\mu$ is a finite Borel measure on the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$ and fix two Borel functions $\phi$ on $\mathbb{R}^d$ and $\psi$ on $\mathbb{S}$, both of which take values in the unit ball of $\mathbb{C}$. We define the associated multiplier $m = m_{\phi,\psi,\mu,\nu}$ on $\mathbb{R}^d$ by

$$m(\xi) = \frac{\frac{1}{2}\int_{\mathbb{S}}\langle \xi, \theta\rangle^2\psi(\theta)\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1-\cos\langle \xi, x\rangle]\phi(x)\nu(\mathrm{d}x)}{\frac{1}{2}\int_{\mathbb{S}}\langle \xi, \theta\rangle^2\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1-\cos\langle \xi, x\rangle]\nu(\mathrm{d}x)} \tag{1.1}$$

if the denominator is not 0, and $m(\xi) = 0$ otherwise. Here $\langle \cdot, \cdot\rangle$ stands for the scalar product in $\mathbb{R}^d$. This class includes many important examples (for the list of these, consult [3] and [4]), for instance it contains the real and imaginary parts of the Beurling-Ahlfors transform $B$ on the complex plane. Recall that the latter operator is a Fourier multiplier with the symbol $m(\xi) = \overline{\xi}/\xi$, $\xi \in \mathbb{C}$. This object is of fundamental importance in the study of partial differential equations and quasiconformal mappings; in particular, it changes the complex derivative $\overline{\partial}$ to $\partial$. By some simple computations, one shows the identity $B = (R_1^2 - R_2^2) + 2iR_1R_2$ which links the Beurling-Ahlfors operator with planar second-order Riesz transforms. Now, it is not difficult to see that both $R_1^2 - R_2^2$ and $2R_1R_2$ can be represented as the Fourier multipliers with the symbols of the form (1.1). Indeed, the choice $d = 2$, $\mu = \delta_{(1,0)} + \delta_{(0,1)}$, $\psi(1, 0) = -1 = -\psi(0, 1)$ and $\nu = 0$ gives rise to $T_m = R_1^2 - R_2^2$; furthermore, $d = 2$, $\mu = \delta_{(1/\sqrt{2},1/\sqrt{2})} + \delta_{(1/\sqrt{2},-1/\sqrt{2})}$, $\psi(1/\sqrt{2}, 1/\sqrt{2}) = 1 = \psi(1/\sqrt{2}, -1/\sqrt{2})$ and $\nu = 0$ leads to $T_m = 2R_1R_2$. For a higher-dimensional example, pick a proper subset $J$ of $\{1, 2, \ldots, d\}$ and take $\mu = \delta_{e_1} + \delta_{e_2} + \ldots + \delta_{e_d}$, $\nu = 0$ and $\psi(e_j) = \chi_J(j)$, $j = 1, 2, \ldots, d$, where $e_1, e_2, \ldots, e_d$ are the versors in $\mathbb{R}^d$. This yields the operator $\sum_{j\in J} R_j^2$ on $\mathbb{R}^d$.

What about the estimates for the Fourier multipliers associated with the symbols from the class (1.1)? One of the principal results of [4] is the following $L_p$ bound.

**Theorem 1.1.** *Let $1 < p < \infty$ and let $m = m_{\phi,\psi,\mu,\nu}$ be given by* (1.1)*. Then for any $f \in L_p(\mathbb{R}^d)$ we have*

$$||T_m f||_{L_p(\mathbb{R}^d)} \leq (p^* - 1)||f||_{L_p(\mathbb{R}^d)}. \tag{1.2}$$

It turns out that the constant $p^* - 1$ is the best possible: see Geiss, Montgomery-Smith and Saksman [16] or Bañuelos and Osękowski [5] for details.

Essentially, all the tight estimates mentioned above are proved with the use of probabilistic methods (Pichorides exploits certain special superharmonic functions, but these, in fact, lead to more general inequalities for orthogonal martingales: see Bañuelos and Wang [6]). It turns out that martingale methods lead to other results for Fourier multipliers (see e.g. [25] and [26]). The purpose of this paper is to continue the study in this direction. The question we plan to address is: what can be said in the limit case $p = 1$ of (1.2)? Clearly, the $L_p$ estimate does not hold for the class (1.1), since it already fails for the Beurling-Ahlfors operator. As shown by the author [25], we have the following $L \log L$ bound: for any $K > 1$ and any Borel subset $A$ of $\mathbb{R}^d$, we have

$$\int_A |T_m f(x)|\mathrm{d}x \leq K \int_{\mathbb{R}^d} \Psi(|f(x)|)\mathrm{d}x + \frac{|A|}{2(K-1)}, \tag{1.3}$$

where $\Psi$ stands for the LlogL function $\Psi(t) = (t+1)\log(t+1) - t$. In this paper we will provide a related *sharp* bound under a different norming of the $L \log L$ space, which is sometimes more convenient to work with (see e.g. the papers [7, 8] by Bennett and the monograph [10] by Bennett and Sharpley). We need more definitions. For a real-, complex-, or Hilbert-space-valued function $\xi$, given on a non-atomic measure space $(X, \mathcal{S}, \eta)$, we define $\xi^*$, the decreasing rearrangement of $\xi$, by

$$\xi^*(t) = \inf\{\lambda \geq 0 : \eta(\{x \in X : |\xi(x)| > \lambda\}) \leq t\}.$$

Then $\xi^{**} : (0, \eta(X)] \to [0, \infty)$, the maximal function of $\xi^*$, is given by the formula

$$\xi^{**}(t) = \frac{1}{t}\int_0^t \xi^*(s)\mathrm{d}s, \qquad t \in (0, \eta(X)).$$

One easily verifies that $\xi^{**}$ can alternatively be defined by

$$\xi^{**}(t) = \frac{1}{t}\sup\left\{\int_E |\xi|\mathrm{d}\eta \,:\, E \in \mathcal{S},\, \eta(E) = t\right\}.$$

Now, given $A \in \mathcal{S}$ with $\eta(A) > 0$, we define the associated space $L \log L(A)$ as the class of all $\xi$ for which

$$||\xi||_{L\log L(A)} := \int_0^{|A|} \xi^{**}(t)\mathrm{d}t < \infty.$$

See [7] and [10] for the list of basic properties of these spaces and much more on the subject. We are ready to formulate the main result of this paper.

**Theorem 1.2.** *Suppose that $m$ is a symbol from the class* (1.1). *Then for any Borel set $A \subset \mathbb{R}^d$ of positive measure and any Borel function $f : \mathbb{R}^d \to \mathbb{C}$ we have*

$$\int_A |T_m f(x)| dx \leq ||f||_{L\log L(A)}. \tag{1.4}$$

*The inequality is sharp even for the real part of the Beurling-Ahlfors transform: for an arbitrary $c < 1$, there is a real-valued function $f$ supported on the unit ball $\mathcal{B}$ of $\mathbb{C}$ such that*

$$\int_{\mathcal{B}} |(R_1^2 - R_2^2)f(x)| dx > c||f||_{L\log L(\mathcal{B})}.$$

It is worth mentioning here that a related statement for the periodic Hilbert transform (conjugate function) was obtained by Bennett in [9]. Namely, if $f$ is an integrable function on the unit circle $\mathbb{T}$ and $\tilde{f}$ denotes the conjugate function to $f$, then

$$\int_{\mathbb{T}} |\tilde{f}(x)|\mathrm{d}x \leq \frac{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots}{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots}||f||_{L\log L(\mathbb{T})}$$

and the constant $(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots)/(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots)=0.7424537\ldots$ is the best possible.

A few words about the organization of the paper is in order. In the next section we study martingale inequalities, which lead to the probabilistic counterpart of (1.4). In Section 3 we combine these estimates with the representation of Fourier multipliers (1.1) in terms of Lévy processes, and provide the proof of the inequality (1.4). In the final section we address the sharpness of our result. This will be based on the technique of laminates, an important object in the convex integration theory.

## 2. Martingale inequalities

The key ingredient of the proof of the estimate (1.4) is an appropriate inequality for differentially subordinated martingales. We begin with introducing the necessary probabilistic background and notation, and start with the discrete-time case. Suppose that $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, filtered by a nondecreasing family $(\mathcal{F}_n)_{n\geq 0}$ of sub-$\sigma$-algebras of $\mathcal{F}$. Let $f=(f_n)_{n\geq 0}$, $g=(g_n)_{n\geq 0}$ be two adapted martingales, taking values in a certain separable Hilbert space $\mathcal{H}$ (we may and will assume that $\mathcal{H}=\ell_2$), with the norm denoted by $|\cdot|$. Let $df=(df_n)_{n\geq 0}$, $dg=(dg_n)_{n\geq 0}$ be the difference sequences of $f$ and $g$, respectively; that is,

$$df_0=f_0, \qquad df_n=f_n-f_{n-1} \qquad \text{for } n\geq 1,$$

and similarly for $dg$. Following Burkholder [12], we say that $g$ is differentially subordinate to $f$, if we have

$$|dg_n|\leq|df_n| \qquad \text{for all } n\geq 0. \tag{2.1}$$

For example, this relation holds true if $g$ is a $\pm 1$-transform of $f$, i.e., there is a predictable sequence $\varepsilon=(\varepsilon_n)_{n\geq 0}$ of signs such that $dg_n=\varepsilon_n df_n$ for all $n$.

Now, let us turn our attention to the continuous-time case. Assume that the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is complete and equip it with a continuous-time filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Let $X$, $Y$ be two adapted martingales taking values in $\mathcal{H}$; as usual, we assume that the processes have right-continuous trajectories with the limits from the left. The symbol $[X,Y]$ will stand for the quadratic covariance process of $X$ and $Y$. See e.g. Dellacherie and Meyer [15] for details in the case when the processes are real-valued, and extend the definition to the vector setting by $[X,Y]=\sum_{k=0}^{\infty}[X^k,Y^k]$, where $X^k$, $Y^k$ are the $k$-th coordinates of $X$, $Y$, respectively. Following Bañuelos and Wang [6] and Wang [31], we say that $Y$ is differentially subordinate to $X$, if the process $([X,X]_t-[Y,Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of $t$. Note that any two martingales $f$, $g$ can be thought of as continuous-time martingales (via $X_t=f_{\lfloor t\rfloor}$, $Y_t=g_{\lfloor t\rfloor}$, $t\geq 0$) and then the above domination means that the process

$$[f,f]_n-[g,g]_n=\left(\sum_{k=0}^{n}(|df_k|^2-|dg_k|^2)\right)$$

is nonnegative and nondecreasing (as a function of $n$). This is equivalent to (2.1) and hence the definition above is consistent with the discrete-time differential subordination.

Now we are ready to formulate the main probabilistic results. Here and below, we use the notation $||X||_p = \sup_{t\geq 0} ||X_t||_p$, $1 \leq p \leq \infty$.

**Theorem 2.1.** *Assume that $X$, $Y$ are $\mathcal{H}$-valued martingales such that $Y$ is differentially subordinate to $X$. Then for any $\lambda \geq 1$,*

$$\sup_{t\geq 0} \mathbb{E}\big(|Y_t| - \lambda ||X||_\infty\big)_+ \leq \frac{e^{1-\lambda}}{4} ||X||_1. \tag{2.2}$$

The proof of this fact rests on the direct application of Burkholder's method: we shall deduce the inequality (2.2) from the existence of a family $\{U_\lambda\}_{\lambda\geq 1}$ of certain special functions defined on the set $S = \{(x, y) \in \mathcal{H} \times \mathcal{H} : |x| \leq 1\}$. In order to simplify the technicalities, we shall combine the technique with an "integration argument", invented in [22] (see also [23]): first we introduce a simple function $u_\infty : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, for which the calculations are relatively easy; then define $U_\lambda$ by integrating $u_\infty$ against appropriate nonnegative kernel. Let

$$u_\infty(x, y) = \begin{cases} 0 & \text{if } |x| + |y| \leq 1, \\ (|y| - 1)^2 - |x|^2 & \text{if } |x| + |y| > 1. \end{cases}$$

We have the following fact (see Lemma 2.2 in [23] for a slightly stronger statement in which the differential subordination is replaced by a less restrictive assumption).

**Lemma 2.2.** *Let $X$, $Y$ be $\mathcal{H}$-valued martingales such that $||X||_2 < \infty$ and $Y$ is differentially subordinate to $X$. Then for any $t \geq 0$ we have*

$$\mathbb{E}u_\infty(X_t, Y_t) \leq 0.$$

Keeping the above fact in mind, we fix $\lambda \geq 1$ and define $U_\lambda : S \to \mathbb{R}$ by

$$U_\lambda(x, y) = \frac{1}{4}\int_1^\lambda r^2 e^{r-\lambda} u_\infty(x/r, y/r)\mathrm{d}r + \frac{e^{1-\lambda}}{4}(|y|^2 - |x|^2)$$

(recall that $S$, the domain of $U_\lambda$, is the strip $\{(x, y) \in \mathcal{H} \times \mathcal{H} : |x| \leq 1\}$). A little calculation shows that the function $U_\lambda$ admits the following explicit formula. If $|x| + |y| \leq 1$, then

$$U_\lambda(x, y) = \frac{e^{1-\lambda}}{4}(|y|^2 - |x|^2).$$

If $1 < |x| + |y| \leq \lambda$, then

$$U_\lambda(x, y) = \frac{1 - |x|}{2} e^{|x|+|y|-\lambda} - \frac{e^{1-\lambda}}{4}.$$

Finally, if $|x| + |y| > \lambda$, then

$$U_\lambda(x, y) = \frac{|y|^2 - |x|^2}{4} + \frac{1-\lambda}{2}|y| + \frac{1 - e^{1-\lambda} + (\lambda - 1)^2}{4}.$$

In what follows, we shall need the following majorization property.

**Lemma 2.3.** *For any $\lambda \geq 1$ and any $(x, y) \in S$ we have*

$$U_\lambda(x, y) \geq (|y| - \lambda)_+ - \frac{e^{1-\lambda}}{4}|x|. \tag{2.3}$$

*Proof.* Of course, it suffices to prove the majorization for $\mathcal{H}=\mathbb{R}$ only. Furthermore, note that $U_\lambda$ satisfies the symmetry condition $U_\lambda(x,y)=U_\lambda(-x,y)=U_\lambda(x,-y)$ for all $(x,y)\in S$; hence we may restrict ourselves to $x$, $y\geq 0$. The next observation is that for each $y$, the function $x\mapsto U_\lambda(x,y)$ is concave: indeed, one easily verifies that both $u_\infty$ and $(x,y)\mapsto |y|^2-|x|^2$ have this property. On the other hand, for any fixed $y$, the right-hand side of (2.3) is convex function of $x\in[0,1]$. Consequently, it is enough to verify the majorization under the additional assumption $x\in\{0,1\}$. If $x=1$ and $y\leq\lambda-1$, then both sides of (2.3) are equal. If $x=1$ and $y>\lambda-1$, the majorization can be rewritten in the equivalent form

$$\frac{(y-\lambda+1)^2}{4}\geq (y-\lambda)_+.$$

This bound is evident when $y\leq\lambda$, while for $\lambda>1$ it can be transformed into $(y-\lambda-1)^2/4\geq 0$, which is also true. Suppose next that $x=0$. If $y=0$, both sides of (2.3) are equal; then, for $y\in(0,\lambda]$, the left-hand side increases, while the right-hand side is constant; thus, the majorization is also valid for this choice of $x$ and $y$. Finally, if $x=0$ and $y>\lambda$, the inequality is equivalent to

$$\frac{(y-\lambda-1)^2}{4}+\frac{1-e^{1-\lambda}}{4}\geq 0,$$

which clearly holds true. This completes the proof. □

Now we are ready to establish Theorem 2.1.

*Proof of* (2.2). By homogeneity, it suffices to show that for each $t\geq 0$ we have

$$\mathbb{E}(|Y_t|-\lambda)_+\leq \frac{e^{1-\lambda}}{4}||X||_1,$$

provided $X$, $Y$ are as in the statement and satisfy the additional condition $||X||_\infty=1$. So, let us fix $t\geq 0$. We have $\mathbb{E}|Y_t|^2=\mathbb{E}[Y,Y]_t\leq\mathbb{E}[X,X]_t\leq\mathbb{E}|X_t|^2\leq 1$, in the light of the differential subordination and the boundedness assumption on $X$. Therefore, Lemma 2.2 and Fubini's theorem imply

$$\mathbb{E}U_\lambda(X_t,Y_t)\leq\frac{1}{4}\int_1^\lambda r^2e^{r-\lambda}\mathbb{E}u_\infty(X_t/r,Y_t/r)\mathrm{d}r\leq 0. \tag{2.4}$$

To see that Fubini's theorem is applicable, note that $|u_\infty(x,y)|\leq c(|x|^2+|y|^2+1)$ for all $x$, $y\in\mathcal{H}$ and some absolute constant $c$; thus

$$\mathbb{E}\int_1^\lambda r^2e^{r-\lambda}|u_\infty(X_t/r,Y_t/r)|\mathrm{d}r\leq\tilde{c}\mathbb{E}(|X_t|^2+|Y_t|^2+1)<\infty,$$

where $\tilde{c}$ is another universal constant. Hence (2.4) is indeed true, and combining this bound with (2.3) yields

$$\mathbb{E}(|Y_t|-\lambda)_+\leq\frac{e^{1-\lambda}}{4}\,\mathbb{E}|X_t|\leq\frac{e^{1-\lambda}}{4}\,||X||_1. \tag{2.5}$$

This is exactly the claim. □

The above theorem implies the following statement, which will be useful in our further considerations.

**Corollary 2.4.** *Assume that $X$, $Y$ are martingales taking values in $\mathcal{H}$ such that $||X||_\infty \leq 1$ and $Y$ is differentially subordinate to $X$. Then for any $\lambda \geq 1$, $t \geq 0$ and any $A \in \mathcal{F}$ we have*

$$\mathbb{E}|Y_t|1_A \leq \frac{e^{1-\lambda}}{4}\mathbb{E}|X_t| + \lambda\mathbb{P}(A). \tag{2.6}$$

*Proof.* Fix $\lambda$, $t$, $A$ as in the statement and consider the decomposition $A = A^- \cup A^+$, where

$$A^- = A \cap \{|Y_t| < \lambda\}, \qquad A^+ = A \cap \{|Y_t| \geq \lambda\}.$$

Then, of course,

$$\mathbb{E}(|Y_t| - \lambda)1_{A^-} \leq 0$$

and

$$\mathbb{E}(|Y_t| - \lambda)1_{A^+} \leq \mathbb{E}(|Y_t| - \lambda)_+ \leq \frac{e^{1-\lambda}}{4}\mathbb{E}|X_t|,$$

where in the last passage we have exploited (2.5). Adding the two estimates above, we get

$$\mathbb{E}(|Y_t| - \lambda)1_A \leq \frac{e^{1-\lambda}}{4}\mathbb{E}|X_t|,$$

and this inequality is equivalent to (2.6). □

Before we proceed to the estimates for Fourier multipliers, let us establish here a sharp $L \log L$ estimate for martingale transforms, which can be regarded as a probabilistic version of Theorem 1.2. Unfortunately, we do not know how to show the bound in the general setting of continuous-time differentially subordinated martingales: in the proof we exploit a duality argument which seems to be available only for martingale transforms. However, we believe that the result deserves to be stated separately. We use the notation $||f||_{L\log L(\Omega)} = \sup_{n\geq 0}||f_n||_{L\log L(\Omega)}$, $||g||_1 = \sup_{n\geq 0}||g_n||_1$ and denote the inner product of $\mathcal{H}$ by $\langle\cdot,\cdot\rangle$.

**Theorem 2.5.** *If $f = (f_n)_{n\geq 0}$ is an $\mathcal{H}$-valued martingale and $g = (g_n)_{n\geq 0}$ is its transform by a predictable sequence $v = (v_n)_{n\geq 0}$ with values in $[-1, 1]$, then*

$$||g||_1 \leq ||f||_{L\log L(\Omega)}. \tag{2.7}$$

*The inequality is sharp: for any $c < 1$ there is a martingale $f = (f_n)_{n\geq 0}$ and its $\pm 1$-transform $g = (g_n)_{n\geq 0}$ such that*

$$||g||_1 > c||X||_{L\log L(\Omega)}.$$

*Proof of* (2.7). We may assume that $||f||_{L\log L(\Omega)} < \infty$, since otherwise there is nothing to prove. Fix a nonnegative integer $n$ and consider the random variable $\eta = g_n/|g_n|$, with the convention $\eta = 0 \in \mathcal{H}$ when $g_n = 0$. Clearly, this variable is bounded by 1 and hence the associated martingale $(\eta_n)_{n\geq 0} = (\mathbb{E}(\eta|\mathcal{F}_n))_{n\geq 0}$ also enjoys this property. Using the orthogonality of the martingale differences and

Hardy-Littlewood inequality $\mathbb{E}\langle \xi, \zeta\rangle \leq \int_0^1 \xi^*(s)\zeta^*(s)\mathrm{d}s$, we may write

$$(2.8)\qquad \begin{aligned} ||g_n||_1 = \mathbb{E}\langle g_n, \eta\rangle &= \mathbb{E}\langle g_n, \eta_n\rangle \\ &= \mathbb{E}\left\langle \sum_{k=0}^n v_k df_k, \sum_{k=0}^n d\eta_k \right\rangle \\ &= \mathbb{E}\sum_{k=0}^n v_k \langle df_k, d\eta_k\rangle \\ &= \mathbb{E}\left\langle \sum_{k=0}^n df_k, \sum_{k=0}^n v_k d\eta_k \right\rangle \\ &\leq \int_0^1 f_n^*(s) \left(\sum_{k=0}^n v_k d\eta_k\right)^*(s)\mathrm{d}s. \end{aligned}$$

However, for any $u \in (0,1]$ we have

$$\int_0^u \left(\sum_{k=0}^n v_k d\eta_k\right)^*(s)\mathrm{d}s = \sup\left\{ \int_A \left|\sum_{k=0}^n v_k d\eta_k\right| \mathrm{d}\mathbb{P} \,:\, \mathbb{P}(A) = u\right\},$$

which, as we will show now, does not exceed $u(1-\log u)$. Suppose first that $u \leq 1/4$. The martingale $(\sum_{k=0}^m v_k d\eta_k)_{m\geq 0}$ is differentially subordinate to $(\eta_m)_{m\geq 0}$, and the latter is bounded by 1. Consequently, by the above corollary,

$$\int_0^u \left(\sum_{k=0}^n v_k d\eta_k\right)^*(s)\mathrm{d}s \leq \inf_{\lambda\geq 1}\left\{\lambda u + \frac{e^{1-\lambda}}{4}\right\}.$$

Now a straightforward analysis shows that the infimum is attained for $\lambda = 1 - \log(4u) \geq 1$ and equal to $u(1-\log(4u)) = u(1-\log u) + u(1-\log 4) \leq u(1-\log u)$. On the other hand, if $u = \mathbb{P}(A) \geq 1/4$, we use Schwarz inequality and a trivial $L^2$ bound for martingale transforms (with constant 1) to obtain

$$\begin{aligned} \int_A \left|\sum_{k=0}^n v_k d\eta_k\right| \mathrm{d}\mathbb{P} &\leq (\mathbb{P}(A))^{1/2} \left|\left|\sum_{k=0}^n v_k d\eta_k\right|\right|_2 \\ &\leq u^{1/2}\left|\left|\sum_{k=0}^n d\eta_k\right|\right|_2 \leq u^{1/2}, \end{aligned}$$

where in the latter passage we have exploited the boundedness of $(\eta_m)_{m\geq 0}$ again. Now it is elementary to prove that $u^{1/2} \leq u(1-\log u)$: indeed, the function $F(u) = u^{1/2}(1-\log u) - 1$ is concave on $[1/4, 1]$ and satisfies $F(1) = 0$, $F(1/4) = \log 2 - 1/2 > 0$. Thus, we have shown that

$$\int_0^u \left(\sum_{k=0}^n v_k d\eta_k\right)^*(s)\mathrm{d}s \leq u(1-\log u) = \int_0^u \log\frac{1}{s}\mathrm{d}s$$

for all $u \in (0,1]$. By Hardy's lemma, this implies that for any nonincreasing function $\xi : (0,1] \to [0,\infty)$ we have

$$\int_0^1 \xi(s)\left(\sum_{k=0}^n v_k d\eta_k\right)^*(s)\mathrm{d}s \leq \int_0^1 \xi(s)\log\frac{1}{s}\mathrm{d}s.$$

Choosing $\xi(s) = f_n^*(s)$ and coming back to (2.8), we see that

$$||g_n||_1 \leq \int_0^1 f_n^*(s) \log \frac{1}{s} \mathrm{d}s = \int_0^1 f_n^{**}(s) \mathrm{d}s = ||f_n||_{L \log L(\Omega)} \leq ||f||_{L \log L(\Omega)}.$$

Taking the supremum over $n$ completes the proof. $\square$

*Sharpness.* Now we will show that the constant 1 cannot be decreased in the martingale bound. Fix a large positive integer $n$ and consider a sequence $\xi_0$, $\xi_1$, $\ldots$, $\xi_{2n+1}$ of independent random variables, with the distribution given by $\xi_0 \equiv 1$ and $\mathbb{P}(\xi_k = -k) = 1 - \mathbb{P}(\xi_k = 1) = (k+1)^{-1}$. Introduce the stopping time $\tau = \inf\{k : \xi_k < 0\}$, with the convention $\inf \emptyset = \infty$. All the variables $\xi_k$, except for the first of them, have mean zero; hence Doob's optional sampling theorem implies that the sequence $f = (f_k)_{k=0}^{2n+1}$, given by

$$f_k = \xi_0 + \xi_1 + \ldots + \xi_{\tau \wedge k}, \qquad k \leq 2n+1,$$

is a martingale adapted to the natural filtration. This sequence behaves as follows: it starts from 1 and, in each step, it either jumps to zero and stays there forever, or increases by 1. Therefore, for any $0 \leq k \leq 2n+1$ we have $\mathbb{P}(f_k = k+1) = (k+1)^{-1} = 1 - \mathbb{P}(f_k = 0)$. This in turn implies $f_k^* = (k+1)\chi_{[0,(k+1)^{-1}]}$, $f_k^{**}(s) = (k+1)\chi_{[0,(k+1)^{-1}]}(s) + s^{-1}\chi_{((k+1)^{-1},1]}(s)$ and, finally,

$$||f_k||_{L \log L(\Omega)} = \int_0^1 f_k^{**}(s) \mathrm{d}s = 1 + \int_{(k+1)^{-1}}^1 s^{-1} \mathrm{d}s = 1 + \log(k+1).$$

Thus, we see that $||f||_{L \log L(\Omega)} = \sup_{0 \leq k \leq 2n+1} ||f_k||_{L \log L(\Omega)} = 1 + \log(2n+2)$. Now, let $g$ be the transform of $f$ by the deterministic sequence $1, -1, 1, -1, \ldots$. It is easy to see that the terminal random variable $|g_{2n+1}|$ takes values in the set $\{0, 2, 4, 6, \ldots, 2n+2\}$. More precisely, for any $k \in \{1, 2, \ldots, n\}$ we have

$$\begin{aligned}
\mathbb{P}(|g_{2n+1}| = 2k) &= \mathbb{P}(\tau = 2k-1) + \mathbb{P}(\tau = 2k) \\
&= \mathbb{P}(\xi_0 = \xi_1 = \ldots = \xi_{2k-2} = 1,\, \xi_{2k-1} = 1 - 2k) \\
&\quad + \mathbb{P}(\xi_0 = \xi_1 = \ldots = \xi_{2k-1} = 1,\, \xi_{2k} = -2k) \\
&= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2k-2}{2k-1} \cdot \frac{1}{2k} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2k-1}{2k} \cdot \frac{1}{2k+1} \\
&= \frac{1}{2k} \left( \frac{1}{2k-1} + \frac{1}{2k+1} \right).
\end{aligned}$$

One can compute similarly the probabilities $\mathbb{P}(|g_{2n+1}| = 0)$ and $\mathbb{P}(|g_{2n+1}| = 2n+1)$, but we will not need them. From the above formulas, we infer that

$$||g_{2n+1}||_1 \geq \sum_{k=1}^n \left( \frac{1}{2k-1} + \frac{1}{2k+1} \right),$$

and the latter expression is of order $\log(2n)$. Hence, the above bound implies

$$\liminf_{n \to \infty} \frac{||g_{2n+1}||_1}{||f_{2n+1}||_{L \log L}} \geq 1,$$

and the desired sharpness follows. $\square$

## 3. $L\log L$ inequality for Fourier multipliers

Let us now describe the machinery which can be used to deduce inequalities for Fourier multipliers from the corresponding results from the martingale theory. Let $m = m_{\phi,\psi,\mu,\nu}$ be a multiplier as in (1.1). By the results in [4], we may assume that the Lévy measure $\nu$ satisfies the symmetry condition $\nu(B) = \nu(-B)$ for all Borel subsets $B$ of $\mathbb{R}^d$. More precisely, there are $\bar{\mu}$, $\bar{\nu}$, $\bar{\phi}$, $\bar{\psi}$ such that $\bar{\nu}$ is symmetric and $m_{\phi,\psi,\mu,\nu} = m_{\bar{\phi},\bar{\psi},\bar{\mu},\bar{\nu}}$. Assume in addition that $|\nu| = \nu(\mathbb{R}^d)$ is finite and nonzero, and define $\tilde{\nu} = \nu/|\nu|$. Consider the independent random variables $T_{-1}$, $T_{-2}$, $\ldots$, $Z_{-1}$, $Z_{-2}$, $\ldots$ such that for each $n = -1, -2, \ldots$, $T_n$ has exponential distribution with parameter $|\nu|$ and $Z_n$ takes values in $\mathbb{R}^d$ and has $\tilde{\nu}$ as the distribution. Next, put $S_n = -(T_{-1} + T_{-2} + \ldots + T_n)$ for $n = -1, -2, \ldots$ and let

$$X_{s,t} = \sum_{s<S_j\leq t} Z_j, \qquad X_{s,t-} = \sum_{s<S_j<t} Z_j, \qquad \Delta X_{s,t} = X_{s,t} - X_{s,t-},$$

for $-\infty < s \leq t \leq 0$. For a given $f \in L_\infty(\mathbb{R}^d)$, define its parabolic extension $\mathcal{U}_f$ to $(-\infty, 0] \times \mathbb{R}^d$ by

$$\mathcal{U}_f(s,x) = \mathbb{E}f(x + X_{s,0}).$$

Next, fix $x \in \mathbb{R}^d$, $s < 0$ and $f \in L_\infty(\mathbb{R}^d)$. We introduce the processes $F = (F_t^{x,s,f})_{t\in[s,0]}$ and $G = (G_t^{x,s,f,\phi})_{t\in[s,0]}$ by

$$\begin{aligned} F_t &= \mathcal{U}_f(t, x + X_{s,t}), \\ G_t &= \sum_{s<u\leq t} \big[\Delta F_u \cdot \phi(\Delta X_{s,u})\big] \\ &\quad - \int_s^t \int_{\mathbb{R}^d} \big[\mathcal{U}_f(v, x + X_{s,v-} + z) - \mathcal{U}_f(v, x + X_{s,v-})\big]\phi(z)\nu(\mathrm{d}z)\mathrm{d}v. \end{aligned} \tag{3.1}$$

Note that the sum in the definition of $G$ can be seen as the result of modulating of the jumps of $F$ by $\phi$, and the subsequent double integral can be regarded as an appropriate compensator. We have the following statement, proved in [3].

**Lemma 3.1.** *For any fixed $x$, $s$, $f$ as above, the processes $F^{x,s,f}$, $G^{x,s,f,\phi}$ are martingales with respect to $(\mathcal{F}_t)_{t\in[s,0]}$. Furthermore, if $||\phi||_\infty \leq 1$, then $G^{x,s,f,\phi}$ is differentially subordinate to $F^{x,s,f}$.*

Now, fix $s < 0$ and define the operator $\mathcal{S} = \mathcal{S}^{s,\phi,\nu}$ by the bilinear form

$$\int_{\mathbb{R}^d} \mathcal{S}f(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\big[G_0^{x,s,f,\phi} g(x + X_{s,0})\big]\mathrm{d}x, \tag{3.2}$$

where $f$, $g \in C_0^\infty(\mathbb{R}^d)$. By a simple density argument, we can show that this identity holds for all $f \in C_0^\infty(\mathbb{R}^d)$ and all $g \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ (see Lemma 3.2 in [24] for a similar reasoning). The following fact, proved in [3], represents the above operators as Fourier multipliers.

**Lemma 3.2.** *Let $1 < p < \infty$ and $d \geq 2$. The operator $\mathcal{S}^{s,\phi,\nu}$ is well defined and extends to a bounded operator on $L_p(\mathbb{R}^d)$, which can be expressed as a Fourier multiplier with the symbol*

$$\begin{aligned} M(\xi) &= M_{s,\phi,\nu}(\xi) \\ &= \left[1 - \exp\left(2s \int_{\mathbb{R}^d} (1 - \cos\langle \xi, z\rangle)\nu(dz)\right)\right] \frac{\int_{\mathbb{R}^d} (1 - \cos\langle \xi, z\rangle)\phi(z)\nu(dz)}{\int_{\mathbb{R}^d} (1 - \cos\langle \xi, z\rangle)\nu(dz)} \end{aligned}$$

*if* $\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz)\neq 0$, *and* $M(\xi)=0$ *otherwise.*

Our first step in the analysis of the estimate (1.4) is to establish the following counterpart of Theorem 2.1.

**Theorem 3.3.** *Let* $m$ *be a symbol belonging to the class* (1.1) *and take* $f\in L_1(\mathbb{R}^d)\cap L_\infty(\mathbb{R}^d)$ *satisfying* $||f||_{L_\infty(\mathbb{R}^d)}\leq 1$. *Then for any* $\lambda\geq 1$, $t\geq 0$ *and any* $A\in\mathcal{F}$ *we have*

$$\int_A |T_m f(x)|dx \leq \frac{e^{1-\lambda}}{4}||f||_{L^1(\mathbb{R}^d)}+\lambda|A|. \tag{3.3}$$

*Proof.* We have split the reasoning into a few intermediate parts.

*Step 1.* First we show the estimate for the multipliers of the form

$$M_{\phi,\nu}(\xi)=\frac{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\phi(z)\nu(\mathrm{d}z)}{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(\mathrm{d}z)}, \tag{3.4}$$

with $\nu$ satisfying $0<\nu(\mathbb{R}^d)<\infty$ (this assumption makes the above machinery using Lévy processes applicable). Fix $s<0$, a function $f\in C_0^\infty(\mathbb{R}^d)$ taking values in the unit ball of $\mathbb{C}$ and a function $g$ on $\mathbb{R}^d$, supported on the set $A$ and also taking values in the unit ball of $\mathbb{C}$. Of course, then the associated martingale $F^{x,s,f}$ is bounded by 1. By Fubini's theorem and (2.6), we have, for any $\lambda\geq 1$,

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\mathbb{E}\big[G_0^{x,s,f,\phi}g(x+X_{s,0})\big]\mathrm{d}x\right| &\leq \int_{\mathbb{R}^d}\mathbb{E}\big|G_0^{x,s,f,\phi}\big|1_{\{x+X_{s,0}\in A\}}\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d}\left\{\frac{e^{1-\lambda}}{4}\mathbb{E}|F_0^{x,s,f}|+\lambda\mathbb{P}(x+X_{s,0}\in A)\right\}\mathrm{d}x\\ &=\frac{e^{1-\lambda}}{4}||f||_{L_1(\mathbb{R}^d)}+\lambda|A|.\end{aligned}$$

Plugging this into the definition of $\mathcal{S}$ and taking the supremum over all $g$ as above, we obtain

$$\int_A|\mathcal{S}^{s,\phi,\nu}f(x)|\mathrm{d}x\leq\frac{e^{1-\lambda}}{4}||f||_{L_1(\mathbb{R}^d)}+\lambda|A|. \tag{3.5}$$

Now if we let $s\to-\infty$, then $M_{s,\phi,\nu}$ converges pointwise to the multiplier $M_{\phi,\nu}$ given by (3.4). Thus, by Plancherel's theorem, we have $\mathcal{S}^{s,\phi,\nu}f\to T_{M_{\phi,\nu}}f$ in $L_2$ and hence there is a sequence $(s_n)_{n=1}^\infty$ converging to $-\infty$ such that $\mathcal{S}^{s_n,\phi,\nu}f\to T_{M_{\phi,\nu}}f$ almost everywhere. So, Fatou's lemma combined with (3.5) yields the bound

$$\int_A|T_{M_{\phi,\nu}}f(x)|\mathrm{d}x\leq\frac{e^{1-\lambda}}{4}||f||_{L_1(\mathbb{R}^d)}+\lambda|A|. \tag{3.6}$$

*Step 2.* Now we deduce the result for the general multipliers as in (1.1) and drop the assumption $0<\nu(\mathbb{R}^d)<\infty$. For a given $\varepsilon>0$, define a Lévy measure $\nu_\varepsilon$ in polar coordinates $(r,\theta)\in(0,\infty)\times\mathbb{S}$ by

$$\nu_\varepsilon(\mathrm{d}r\mathrm{d}\theta)=\varepsilon^{-2}\delta_\varepsilon(\mathrm{d}r)\mu(d\theta).$$

Here $\delta_\varepsilon$ denotes Dirac measure on $\{\varepsilon\}$. Next, consider a multiplier $M_{\varepsilon,\phi,\psi,\mu,\nu}$ as in (3.4), in which the Lévy measure is $1_{\{|x|>\varepsilon\}}\nu+\nu_\varepsilon$ and the jump modulator is given by $1_{\{|x|>\varepsilon\}}\phi(x)+1_{\{|x|=\varepsilon\}}\psi(x/|x|)$. Note that this Lévy measure is finite and nonzero, at least for sufficiently small $\varepsilon$. If we let $\varepsilon\to 0$, we see that

$$\begin{aligned}\int_{\mathbb{R}^d}[1-\cos\langle\xi,x\rangle]\psi(x/|x|)\nu_\varepsilon(\mathrm{d}x)&=\int_{\mathbb{S}}\langle\xi,\theta\rangle^2\phi(\theta)\frac{1-\cos\langle\xi,\varepsilon\theta\rangle}{\langle\xi,\varepsilon\theta\rangle^2}\mu(d\theta)\\&\to\frac{1}{2}\int_{\mathbb{S}}\langle\xi,\theta\rangle^2\phi(\theta)\mu(\mathrm{d}\theta)\end{aligned}$$

and, consequently, $M_{\varepsilon,\phi,\psi,\mu,\nu}\to m_{\phi,\psi,\mu,\nu}$ pointwise. Thus (3.6) yields (3.3). Indeed, using Plancherel's theorem as above, we see that there is a sequence $(\varepsilon_n)_{n\geq 1}$ converging to 0 such that $T_{M_{\varepsilon_n,\phi,\psi,\mu,\nu}}f\to T_{m_{\phi,\psi,\mu,\nu}}f$ almost everywhere. It suffices to apply Fatou's lemma. □

We turn our attention to the proof of our main estimate.

*Proof of* (1.4). Let $g(x)=T_mf(x)/|T_mf(x)|$ (if $T_mf(x)=0$, set $g(x)=0$ as well). We have

$$\begin{aligned}\int_A|T_mf(x)|\mathrm{d}x&=\int_{\mathbb{R}^d}\langle T_mf(x),g(x)\chi_A(x)\rangle\mathrm{d}x\\&=\int_{\mathbb{R}^d}\langle f(x),T_{\bar{m}}(g\chi_A)(x)\rangle\mathrm{d}x\leq\int_0^\infty f^*(s)(T_{\bar{m}}(g\chi_A))^*(s)\mathrm{d}s.\end{aligned}\tag{3.7}$$

For any positive number $u$, we may write

$$\int_0^u(T_{\bar{m}}(g\chi_A))^*(s)\mathrm{d}s=\sup\left\{\int_E|T_{\bar{m}}(g\chi_A)(x)|\mathrm{d}x\,:\,|E|=u\right\}.$$

Now, assume that $4u=4|E|\leq|A|$. If $m$ belongs to the class (1.1), then so does $\bar{m}$ (one needs to replace $\phi$, $\psi$ by $\bar{\phi}$, $\bar{\psi}$) and hence, by (3.3),

$$\int_E|T_{\bar{m}}(g\chi_A)(x)|\mathrm{d}x\leq\frac{e^{1-\lambda}}{4}||g\chi_A||_{L^1(\mathbb{R}^d)}+\lambda|E|\leq\frac{e^{1-\lambda}}{4}|A|+\lambda|E|.$$

Let us minimize the right-hand side over $\lambda$; one easily checks that the minimum is attained for the choice $\lambda=1+\log(|A|/(4|E|))$ (note that this value is at least 1, so the application of (3.6) is permitted). We obtain the estimate

$$\begin{aligned}\int_E|T_{\bar{m}}(g\chi_A)(x)|\mathrm{d}x&\leq\left(2+\log\frac{|A|}{4|E|}\right)|E|\\&\leq\left(1+\log\frac{|A|}{u}\right)u=\int_0^u\log\frac{|A|}{s}\mathrm{d}s.\end{aligned}\tag{3.8}$$

On the other hand, if $4u=4|E|>|A|$, then we apply Schwarz inequality and the $L_2$ bound (1.2) for $T_{\bar{m}}$ to obtain

$$\int_E|T_{\bar{m}}(g\chi_A)(x)|\mathrm{d}x\leq|E|^{1/2}||T_{\bar{m}}(g\chi_A)||_{L^2(\mathbb{R}^d)}\leq u^{1/2}||g\chi_A||_{L^2(\mathbb{R}^d)}\leq u^{1/2}|A|^{1/2}.$$

But this is not larger than $(1+\log\frac{|A|}{u})u$; we have already shown this inequality in the proof of the martingale bound. Consequently, the estimate

$$\int_E|T_{\bar{m}}(g\chi_A)(x)|\mathrm{d}x\leq\left(1+\log\frac{|A|}{u}\right)u=\int_0^u\log\frac{|A|}{s}\mathrm{d}s$$

holds for all $u$. Hence

$$\int_0^u(T_{\bar{m}}(g\chi_A))^*(s)\mathrm{d}s\leq\int_0^u\log\frac{|A|}{s}\mathrm{d}s$$

is true for all $u$. By Hardy's lemma, for any nonincreasing function $\xi$ on $[0,\infty)$,

$$\int_0^\infty \xi(s)(T_{\tilde{m}}(g\chi_A))^*(s)\mathrm{d}s \le \int_0^\infty \xi(s)\log\frac{|A|}{s}\mathrm{d}s.$$

We choose $\xi(s)=f^*(s)$ and combine it with (3.7) to obtain

$$\int_A |T_m f(x)|\mathrm{d}x \le \int_0^\infty f^*(s)\log\frac{|A|}{s}\mathrm{d}s \le \int_0^\infty f^*(s)\log^+\frac{|A|}{s}\mathrm{d}s = ||f||_{L\log L(A)}.$$

This completes the proof of the desired bound. □

In the remainder of this section we discuss the possibility of extending (1.4) to the case of vector-valued multipliers. For any bounded function $m=(m_1,m_2,\ldots,m_n):\mathbb{R}^d\to\mathbb{C}^n$, we may define the associated Fourier multiplier acting on complex valued functions on $\mathbb{R}^d$ by the formula $T_m f=(T_{m_1}f,T_{m_2}f,\ldots,T_{m_n}f)$. The reasoning presented above can be easily modified to yield the following statement.

**Theorem 3.4.** *Let $\nu$, $\mu$ be two measures on $\mathbb{R}^d$ and $\mathbb{S}$, respectively, as in the definition of class* (1.1)*. Assume further that $\phi$, $\psi$ are two Borel functions on $\mathbb{R}^d$ taking values in the unit ball of $\mathbb{C}^n$ and let $m:\mathbb{R}^d\to\mathbb{C}^n$ be the associated symbol given by* (1.1)*. Then for any complex valued function $f$ on $\mathbb{R}^d$, any $\lambda\ge 1$ and any Borel set $A\subset\mathbb{R}^d$ with $|A|>0$ we have*

$$\int_A |T_m f(x)|dx \le \frac{e^{1-\lambda}}{4}||f||_{L^1(\mathbb{R}^d)}+\lambda|A| \tag{3.9}$$

*and*

$$\int_A |T_m f(x)|dx \le ||f||_{L\log L(A)}. \tag{3.10}$$

*Proof.* Essentially, the proof is the same as in the complex-valued setting. Given a $C^\infty$ function $f:\mathbb{R}^d\to\mathbb{C}$ bounded by 1, we consider the associated complex-valued martingale $F$ and the $\mathbb{C}^n$-valued martingale $G=(G^1,G^2,\ldots,G^n)$, defined by the formula (3.1). It is easy to check that $G$ is differentially subordinate to $F$, arguing as in [3] or [4]. Applying the representation (3.2) to each coordinate of $G$ separately, we obtain the associated multiplier $\mathcal{T}=(\mathcal{T}^1,\mathcal{T}^2,\ldots,\mathcal{T}^n)$, where $\mathcal{T}^j$ has symbol $M_{\phi_j,\nu}$ defined in (3.4). Now we repeat the reasoning leading to (3.6), with a vector valued function $g:\mathbb{R}^d\to\mathbb{C}^n$ (the expression $G_0^{x,s,f,\phi}g(x+X_{s,0})$ appearing in the considerations need to be replaced with the corresponding scalar product) and obtain (3.9) with the special multiplier $M_{\phi,\nu}$. The passage to general $m$ as in (1.1) is carried over in the same manner as in the scalar case. The proof of (3.10) follows by the same duality argument as in the complex-valued case. We omit the straightforward details, leaving them to the reader. □

## 4. Lower bounds for the constants

In the final part of the paper we show that the constant 1 appearing on the right-hand side of (1.4) is optimal. Our approach will be based on the properties of certain special probability measures, the so-called laminates. For the sake of convenience and clarity, we have decided to split this section into a few separate parts.

4.1. **Necessary definitions.** Let $\mathbb{R}^{m\times n}$ denote the space of all real matrices of dimension $m\times n$ and let $\mathbb{R}^{n\times n}_{sym}$ be the class of all real symmetric $n\times n$ matrices.

**Definition 4.1.** A function $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ is said to be *rank-one convex*, if $t\mapsto f(A+tB)$ is convex for all $A,B\in\mathbb{R}^{m\times n}$ with rank $B=1$.

Let $\mathcal{P}=\mathcal{P}(\mathbb{R}^{m\times n})$ stand for the class of all compactly supported probability measures on $\mathbb{R}^{m\times n}$. For $\nu\in\mathcal{P}$, we denote by $\overline{\nu}=\int_{\mathbb{R}^{m\times n}}Xd\nu(X)$ the center of mass or *barycenter* of $\nu$.

**Definition 4.2.** We say that a measure $\nu\in\mathcal{P}$ is a *laminate* (and denote it by $\nu\in\mathcal{L}$), if

$$f(\overline{\nu})\leq\int_{\mathbb{R}^{m\times n}}fd\nu \tag{4.1}$$

for all rank-one convex functions $f$. The set of laminates of barycenter 0 is denoted by $\mathcal{L}_0(\mathbb{R}^{m\times n})$.

Laminates arise naturally in several applications of convex integration, where can be used to produce interesting counterexamples, see e.g. [2], [13], [20], [21] and [30]. We will be particularly interested in the case of $2\times 2$ symmetric matrices. The important fact is that laminates can be regarded as probability measures that record the distribution of the gradients of smooth maps, see Corollary 4.6 below. Let us briefly explain this; detailed proofs of the statements below can be found for example in [19], [21] and [30].

**Definition 4.3.** Let $U\subset\mathbb{R}^{2\times 2}$ be a given set. Then $\mathcal{PL}(U)$ denotes the class of *prelaminates* generated in $U$, i.e., the smallest class of probability measures on $U$ which

(i) contains all measures of the form $\lambda\delta_A+(1-\lambda)\delta_B$ with $\lambda\in[0,1]$ and satisfying $\operatorname{rank}(A-B)=1$;

(ii) is closed under splitting in the following sense: if $\lambda\delta_A+(1-\lambda)\tilde{\nu}$ belongs to $\mathcal{PL}(U)$ for some $\tilde{\nu}\in\mathcal{P}(\mathbb{R}^{2\times 2})$ and $\mu$ also belongs to $\mathcal{PL}(U)$ with $\overline{\mu}=A$, then also $\lambda\mu+(1-\lambda)\tilde{\nu}$ belongs to $\mathcal{PL}(U)$.

It follows immediately from the definition that the class $\mathcal{PL}(U)$ contains atomic measures only. Also, by a successive application of Jensen's inequality, we have the inclusion $\mathcal{PL}\subset\mathcal{L}$. Let us state two well-known facts (see [2], [19], [21], [30]).

**Lemma 4.4.** *Let $\nu=\sum_{i=1}^N\lambda_i\delta_{A_i}\in\mathcal{PL}(\mathbb{R}^{2\times 2}_{sym})$ with $\overline{\nu}=0$. Moreover, let $0<r<\frac{1}{2}\min|A_i-A_j|$ and $\delta>0$. For any bounded domain $\Omega\subset\mathbb{R}^2$ there exists $u\in W^{2,\infty}_0(\Omega)$ such that $\|u\|_{C^1}<\delta$ and for all $i=1,\,2,\,\ldots N$,*

$$\left|\{x\in\Omega:\,|D^2u(x)-A_i|<r\}\right|=\lambda_i|\Omega|.$$

**Lemma 4.5.** *Let $K\subset\mathbb{R}^{2\times 2}_{sym}$ be a compact convex set and $\nu\in\mathcal{L}(\mathbb{R}^{2\times 2}_{sym})$ with $\operatorname{supp}\nu\subset K$. For any relatively open set $U\subset\mathbb{R}^{2\times 2}_{sym}$ with $K\subset\subset U$ there exists a sequence $\nu_j\in\mathcal{PL}(U)$ of prelaminates with $\overline{\nu}_j=\overline{\nu}$ and $\nu_j\overset{*}{\rightharpoonup}\nu$.*

Combining these two lemmas and using a simple mollification, we obtain the following statement, proved by Boros, Shékelyhidi Jr. and Volberg [11]. It links laminates supported on symmetric matrices with second derivatives of functions, and will play a crucial role in our argumentation below. Throughout, $\mathcal{B}$ will denote the unit ball in $\mathbb{R}^2$.

**Corollary 4.6.** *Let $\nu \in \mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})$. Then there exists a sequence $u_j \in C_0^\infty(\mathcal{B})$ with uniformly bounded second derivatives, such that*

$$\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} \phi(D^2 u_j(x))\,dx \to \int_{\mathbb{R}^{2\times 2}_{sym}} \phi\, d\nu$$

*for all continuous $\phi : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}$.*

4.2. **Biconvex functions and a special laminate.** In the next step in our analysis, we introduce a certain special laminate. To do this, we need some additional notation. A function $\zeta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is said to be *biconvex* if for any fixed $z \in \mathbb{R}$, the functions $x \mapsto \zeta(x, z)$ and $y \mapsto \zeta(z, y)$ are convex. Now, let $f$, $g$ be martingales of Section 2, which exhibit the sharpness of (2.2) (let $n$, corresponding to the length of these martingales, be fixed). Consider the $\mathbb{R}^2$-valued martingale

$$(F, G) := \left(\frac{f+g}{2}, \frac{f-g}{2}\right).$$

This sequence has the following *zigzag* property: for any $0 \leq k \leq 2n+1$ we have $F_k = F_{k+1}$ with probability 1 or $G_k = G_{k+1}$ almost surely; that is, in each step $(F, G)$ moves either vertically, or horizontally. Indeed, this follows directly from the construction that for each $k$ we have $\mathbb{P}(df_k = dg_k) = 1$ or $\mathbb{P}(df_k = -dg_k) = 1$. This property combines nicely with biconvex functions: if $\zeta$ is such a function, then a successive application of Jensen's inequality gives

$$\mathbb{E}\zeta(F_{2n+1}, G_{2n+1}) \geq \mathbb{E}\zeta(F_{2n}, G_{2n}) \geq \ldots \geq \mathbb{E}\zeta(F_0, G_0) = \zeta(1, 0). \tag{4.2}$$

Now, the martingale $(F, G)$, or rather the distribution of its terminal variable $(F_{2n+1}, G_{2n+1})$, gives rise to a probability measure $\nu$ on $\mathbb{R}^{2\times 2}_{sym}$: put

$$\nu\,(\mathrm{diag}(x, y)) = \mathbb{P}\big((F_{2n+1}, G_{2n+1}) = (x, y)\big), \qquad (x, y) \in \mathbb{R}^2.$$

Here and below, $\mathrm{diag}(x, y)$ denotes the diagonal matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. The key observation is that $\nu$ is a laminate of barycenter $(1, 0)$ (actually, it can be shown that it is even a prelaminate, but we will not need this). To prove this property, note that if $\psi : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is rank-one convex, then $(x, y) \mapsto \psi(\mathrm{diag}(x, y))$ is biconvex and thus, by (4.2),

$$\int_{\mathbb{R}^{2\times 2}} \psi \mathrm{d}\nu = \mathbb{E}\psi(\mathrm{diag}(F_{2n+1}, G_{2n+1})) \geq \psi(\mathrm{diag}(1, 0)) = \psi(\bar{\nu}).$$

To apply Corollary 4.6, we need to modify the laminate so that it has barycenter $0 \in \mathbb{R}^{2\times 2}$. To obtain this, we use a simple symmetrization argument. Namely, it is clear that the measure $\tilde{\nu}$ on $\mathbb{R}^{2\times 2}$, given by $\tilde{\nu}(A) = \nu(-A)$ is also a laminate, this time with barycenter $\mathrm{diag}(-1, 0)$. Consequently, the average $\mu = (\nu + \tilde{\nu})/2$ is a laminate of barycenter $\mathrm{diag}(0, 0)$. Of course, this measure can be interpreted in the above martingale language. Namely, $\tilde{\nu}$ can be obtained by repeating the above construction to the martingales $-f$, $-g$; hence $\mu$ can be alternatively defined by splitting the probability space into two halves, copying $f$, $g$ into one half and $-f$, $-g$ into the other, and then considering the distribution of the appropriate terminal variable.

Now, fix $\varepsilon \in (0, 1)$. Let $u_j \in C_0^\infty(\mathcal{B})$ be a sequence of functions with uniformly bounded second derivatives (say, by a number $M$), approximating $\mu$ in the sense of Corollary 4.6. Let $\phi : \mathbb{R}^{2\times 2}_{sym} \to [0, 2n+2]$ be a continuous function satisfying

$\phi(A) = 2n+2$ when $|A_{11}+A_{22}| = 2n+2$ and $\phi(A) = 0$ when $||A_{11}+A_{22}|-2n-2| \geq \varepsilon$. Here we have used the notation $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. Then

$$\int_{\mathbb{R}^{2\times 2}_{sym}} \phi \mathrm{d}\mu = \mathbb{E} f_{2n+1} = 1$$

and hence, if $j$ is sufficiently large, then $\left| \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} \phi(D^2 u_j(x)) \mathrm{d}x - 1 \right| \leq \varepsilon$. But, by the very definition, we have

$$\phi(A) \leq \begin{cases} 2n+2 & \text{if } ||A_1 + A_2| - 2n - 2| \leq \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

so the previous estimate gives $\frac{1}{|\mathcal{B}|}|\{x \in \mathcal{B} : ||\Delta u_j(x)|-2n-2| \leq \varepsilon\}| \geq (1-\varepsilon)/(2n+2)$. A similar argument, exploiting the function $\phi : \mathbb{R}^{2\times 2}_{sym} \to [0,1]$ such that $\phi(A) = 1$ when $|A_{11} + A_{12}| < \varepsilon/2$ and $\phi(A) = 0$ when $|A_{11} + A_{12}| > \varepsilon$, shows that if $j$ is sufficiently large, then

$$\frac{1}{|\mathcal{B}|}|\{x \in \mathcal{B} : |\Delta u_j(x)| \leq \varepsilon\}| \geq (1-\varepsilon) \cdot (1 - (2n+2)^{-1}).$$

Consequently, we see that $(\Delta u_j)^*$ satisfies

$$(\Delta u_j)^*(s) \leq \begin{cases} 2M & \text{if } s \leq \varepsilon|\mathcal{B}|, \\ 2n+2+\varepsilon & \text{if } \varepsilon|\mathcal{B}| < s \leq \varepsilon|\mathcal{B}| + \frac{1-\varepsilon}{2n+2}|\mathcal{B}|, \\ \varepsilon & \text{if } \varepsilon|\mathcal{B}| + \frac{1-\varepsilon}{2n+2}|\mathcal{B}| < s \leq |\mathcal{B}|, \\ 0 & \text{if } s > |\mathcal{B}| \end{cases}$$

and hence

$$\begin{aligned}
&\frac{1}{|\mathcal{B}|}||\Delta u_j||_{L\log L(\mathcal{B})} \\
&= \int_0^1 (\Delta u_j)^*(s|\mathcal{B}|) \log \frac{1}{s} \mathrm{d}s \\
&\leq \int_0^{\varepsilon} 2M \log \frac{1}{s} \mathrm{d}s + \int_{\varepsilon}^{\varepsilon+(1-\varepsilon)/(2n+2)} (2n+2+\varepsilon) \log \frac{1}{s} \mathrm{d}s + \int_{\varepsilon+(1-\varepsilon)/(2n+2)}^1 \varepsilon \log \frac{1}{s} \mathrm{d}s \\
&\leq 2M \int_0^{\varepsilon} \log \frac{1}{s} \mathrm{d}s + (2n+2+\varepsilon) \int_0^{\varepsilon+(2n+2)^{-1}} \log \frac{1}{s} \mathrm{d}s + \varepsilon \int_0^1 \log \frac{1}{s} \mathrm{d}s \\
&= 2M\varepsilon(1 - \log \varepsilon) + (2n+2+\varepsilon)(\varepsilon + (2n+2)^{-1})\big(1 - \log(\varepsilon + (2n+2)^{-1})\big) + \varepsilon.
\end{aligned}$$

For brevity, denote the latter expression by $C(n, \varepsilon)$. Consider the continuous function $\phi(A) = |A_{11} - A_{22}|$ on $\mathbb{R}^{2\times 2}_{sym}$. We have $\int_{\mathbb{R}^{2\times 2}_{sym}} \phi \mathrm{d}\mu = \mathbb{E}|g_{2n+1}| = ||g_{2n+1}||_1$. If $j$ is sufficiently large, then by Corollary 4.6,

$$\frac{\int_{\mathcal{B}} \phi(D^2 u_j(x)) \mathrm{d}x}{|\mathcal{B}| \cdot ||g_{2n+1}||_1} \geq 1 - \varepsilon.$$

Now we put $h = \Delta u_j$ and combine all the above facts together to obtain

$$\begin{aligned}\frac{\int_{\mathcal{B}} |(R_1^2 - R_2^2)h(x)|\mathrm{d}x}{||h||_{L\log L(\mathcal{B})}} &= \frac{\int_{\mathcal{B}} \phi(D^2 u_j(x))\mathrm{d}x}{||\Delta u_j||_{L\log L(\mathcal{B})}} \\ &\geq \frac{(1-\varepsilon)|\mathcal{B}|||g_{2n+1}||_1}{|\mathcal{B}|C(n,\varepsilon)}.\end{aligned}$$

However, $\varepsilon$ was an arbitrary positive number; letting it go to 0, we see that the latter expression converges to

$$\frac{||g_{2n+1}||_1}{1+\log(2n+2)} = \frac{||g_{2n+1}||_1}{||f_{2n+1}||_1}.$$

As we have seen in Section 2, this can be made arbitrarily close to 1 by choosing $n$ sufficiently large. This proves the desired sharpness.

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