SHARP INEQUALITIES FOR HILBERT TRANSFORM IN A VECTOR-VALUED SETTING

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Abstract. The paper is devoted to the study of the periodic Hilbert transform $\mathcal{H}$ in the vector valued setting. Precisely, for any positive integer N we determine the norm of $\mathcal{H}$ as an operator from $L^\infty(\mathbb{T};\ell^N_\infty)$ to $L^p(\mathbb{T};\ell^N_\infty)$, $1 \leq p < \infty$, and from $L^p(\mathbb{T};\ell^N_1)$ to $L^1(\mathbb{T};\ell^N_1)$, for $1 < p \leq \infty$. The proof rests on the existence of a certain family of special harmonic functions.

1. Introduction

The motivation for the results obtained in this paper comes from a very natural question about the periodic Hilbert transform and its action on vector-valued functions. Let us start with some historical perspective. Consider the trigonometric polynomial

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{N} \big(a_k \cos k\theta + b_k \sin k\theta\big), \qquad \theta \in \mathbb{T} \simeq [-\pi, \pi],$$

where a_0, a_1, …, a_N, b_1, b_2, …, b_N are real coefficients. Then the polynomial conjugate to f (or the periodic Hilbert transform of f) is given by

$$\mathcal{H}f(\theta) = \sum_{k=1}^{N} (a_k \sin k\theta - b_k \cos k\theta).$$

A natural question, which interested many mathematicians during the first half of 20th century, can be roughly stated as follows: how does the size of f control the size of its conjugate? Here the sizes of the polynomials can be measured, for instance, in terms of the usual L^p-norms with respect to the normalized Haar measure:

$$||f||_p = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(\theta)|^p \mathrm{d}\theta\right)^{1/p}$$

when $1 \leq p < \infty$, and $||f||_\infty = \operatorname{esssup}_{\theta\in\mathbb{T}} |f(\theta)|$. By the orthogonality of the trigonometric system, we immediately see that $||\mathcal{H}f||_2 \leq ||f||_2$, and this estimate is clearly sharp (the equality can be attained for some nontrivial choice of f). Concerning other values of p, Marcel Riesz [23], [24] proved that

$$||\mathcal{H}f||_p \leq C_p||f||_p, \qquad 1 < p < \infty, \tag{1.1}$$

for some constant C_p which depends only on p (and not on the coefficients of the polynomial f); furthermore, when $p = 1$ or $p = \infty$, then the corresponding estimate does not hold with any finite constant. The optimal value of C_p was determined

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almost 50 years later by Pichorides [22] (see also Gokhberg and Krupnik [17]): the optimal choice turns out to be $\max\{\cot(\pi/2p), \tan(\pi/2p)\}$, $1 < p < \infty$.

The above results of Riesz, together with his related works on interpolation, have had a profound influence on the shape of contemporary analysis and have been extended in numerous directions. In particular, ten years after the discovery of (1.1), its validity for vector-valued functions began to be considered. More precisely, what can be said about the constants in (1.1) if we allow the coefficients a_i, b_j to take values in a certain Banach space $\mathbb{B}$? For example, as Bochner and Taylor showed in [10], the inequality (1.1) does not hold for any p if f is assumed to take values in ℓ^1 or ℓ^∞. One can also study this question from the perspective of other similar estimates (e.g., logarithmic, weak-type, etc.). This problem gained a lot of interest in the literature: see e.g. Aldous [1], Bourgain [4], [5], Burkholder [6]-[8], Calderón and Zygmund [11], McConnell [19] and Rubio de Francia [25], to mention just a few. It turns out that the "good" class of spaces, i.e., those in which (1.1) holds true for all $1 < p < \infty$ with some finite C_p, is that of UMD spaces (Unconditional for Martingale Differences). We will not recall the definition here; for the detailed description, properties and the interplay between these spaces and the estimate (1.1), the reader is referred to the overview article [9] by Burkholder.

The primary goal of this paper is to study a certain version of (1.1), in which the action $L^p \to L^p$ is replaced by $L^\infty \to L^p$ and/or $L^p \to L^1$. Furthermore, we will restrict ourselves to two specific choices of $\mathbb{B}$: ℓ_1^N and ℓ_∞^N. Our main contribution is the identification of the corresponding best constants. The result can be stated as follows.

Theorem 1.1. *Fix a positive integer N.*

(i) For any $1 \leq p < \infty$ and any $f \in L^\infty(\mathbb{T}; \ell_\infty^N)$ we have

$$||\mathcal{H}f||_{L^p(\mathbb{T};\ell_\infty^N)} \leq \left(N \cdot \frac{2^{p+2}}{\pi^{p+1}} \int_0^{\pi/(4N)} (\log \cot s)^p \, ds \right)^{1/p} ||f||_{L^\infty(\mathbb{T};\ell_\infty^N)}. \tag{1.2}$$

(ii) For any $1 < p \leq \infty$ and any $f \in L^p(\mathbb{T}; \ell_1^N)$ we have

$$||\mathcal{H}f||_{L^1(\mathbb{T};\ell_1^N)} \leq \left(N \cdot \frac{2^{p'+2}}{\pi^{p'+1}} \int_0^{\pi/(4N)} (\log \cot s)^{p'} \, ds \right)^{1/p'} ||f||_{L^p(\mathbb{T};\ell_1^N)}, \tag{1.3}$$

where p' denotes the conjugate exponent to p, i.e., $p' = p/(p-1)$.

Both inequalities are sharp.

The problem of finding sharp or almost sharp versions of various estimates for the Hilbert transform has a long history and has been investigated by many mathematicians. See e.g. Davis [12], Essén [14], Essén, Shea and Stanton [15], [16], Gokhberg and Krupnik [17], Janakiraman [18], Osękowski [20], [21], Pichorides [22] and Tomaszewski [26]. We would like to point out that to the best of our knowledge, the above theorem is the first result in the literature which contains the precise information on the constants in the "true" Banach-space setting (all the results cited above concerned the case of real- or Hilbert-space-valued functions). Of course, it would be most desirable to obtain related results for other value spaces (for instance, ℓ_p^N, ℓ_p, or direct sums of such spaces). Unfortunately, this problem seems to be very difficult and not tractable by the methods developed in this paper. On the other hand, we strongly believe that the approach we will introduce below can be applied to wider classes of Fourier multipliers and/or other types of estimates involving ℓ_∞^N and ℓ_1^N-valued functions.

As an application, we obtain sharp exponential and LlogL inequalities for $\mathcal{H}$. Here is the precise statement.

Corollary 1.2. *(i) Suppose that f is a function on $\mathbb{T}$ with values in a unit ball of ℓ_∞^N. Then for any $K<\pi/2$ we have the sharp bound*

$$\frac{1}{2\pi}\int_{\mathbb{T}}\exp\left(K\|\mathcal{H}f(e^{i\theta})\|_{\ell_\infty^N}\right)d\theta\leq\frac{4N}{\pi}\int_{\cot(\pi/(4N))}^\infty\frac{u^{2K/\pi}}{u^2+1}du. \tag{1.4}$$

(ii) Let $\Psi(s)=s\log s-s$, $s\geq 0$. Then for any $f:\mathbb{T}\to\ell_1^N$ with $\int_{\mathbb{T}}\Psi(\|f(e^{i\theta})\|_{\ell_1^N}\|)d\theta<\infty$ and any $L>2/\pi$ we have the sharp inequality

$$\|\mathcal{H}f\|_{L^1(\mathbb{T};\ell_1^N)}\leq\frac{L}{2\pi}\int_{\mathbb{T}}\Psi\left(\|f(e^{i\theta})\|_{\ell_1^N}\right)d\theta+\frac{4LN}{\pi}\int_{\cot(\pi/(4N))}^\infty\frac{u^{2/(\pi L)}}{u^2+1}du. \tag{1.5}$$

A few words about our approach and the organization of the paper are in order. The proof rests on probabilistic methods combined with the existence of a family of certain special harmonic functions on the strip $[-1,1]\times\mathbb{R}$. These special objects are introduced and studied in the next section. Section 3 is devoted to the proofs of the estimates formulated in Theorem 1.1 and Corollary 1.2. The final part of the paper addresses the optimality of the constants involved in these bounds.

2. Special functions and their properties

Suppose that $\lambda\geq 0$ and $p\geq 1$ are fixed numbers. Let $H=\mathbb{R}_+^2=\mathbb{R}\times[0,\infty)$ denote the upper halfplane and introduce the harmonic function $\mathcal{U}_{\lambda,p}:\mathbb{R}\times(0,\infty)\to\mathbb{R}$, given by the Poisson integral

$$\mathcal{U}_{\lambda,p}(\alpha,\beta)=\frac{1}{\pi}\int_{-\infty}^\infty\frac{\beta\left(\left|\frac{2}{\pi}\log|s|\right|^p-\lambda\right)_+}{(\alpha-s)^2+\beta^2}\mathrm{d}s, \tag{2.1}$$

where, as usual, $x_+=\max\{x,0\}$ denotes the positive part of $x\in\mathbb{R}$. It is easy to check that $\mathcal{U}$ satisfies

$$\lim_{(\alpha,\beta)\to(t,0)}\mathcal{U}_{\lambda,p}(\alpha,\beta)=\left(\left|\frac{2}{\pi}\log|t|\right|^p-\lambda\right)_+\qquad\text{for } t\neq 0. \tag{2.2}$$

Next, consider the conformal mapping $\phi(z)=ie^{-i\pi z/2}$, which sends the strip $S=[-1,1]\times\mathbb{R}$ onto H, and define $U_{\lambda,p}:S\to\mathbb{R}$ by the formula

$$U_{\lambda,p}(x,y)=\begin{cases}\mathcal{U}_{\lambda,p}(\phi(x,y)) & \text{if } |x|<1,\\ (|y|^p-\lambda)_+ & \text{if } |x|=1.\end{cases}$$

Clearly, $U_{\lambda,p}$ is harmonic in the interior of S and, by (2.2), it is continuous on this strip. We study the further properties of $U_{\lambda,p}$ in the lemma below. We use the notation $\partial_{xx}U_{\lambda,p}$, $\partial_{xy}U_{\lambda,p}$, etc., for the appropriate second-order partial derivatives of $U_{\lambda,p}$.

Lemma 2.1. *The function $U_{\lambda,p}$ enjoys the following properties.*

(i) $U_{\lambda,p}(x,y)=U_{\lambda,p}(x,-y)=U_{\lambda,p}(-x,y)$ for all $(x,y)\in S$.

(ii) For any $x\in(-1,1)$ and $y\in\mathbb{R}$ we have $\partial_{yy}U_{\lambda,p}(x,y)\geq 0$ and $\partial_{xx}U_{\lambda,p}(x,y)\leq 0$.

(iii) For any $x \in [-1,1]$ we have

$$U_{\lambda,p}(x,0) \leq U_{\lambda,p}(0,0) = \frac{4}{\pi}\int_0^{\pi/4} \left(\left(\frac{2}{\pi}\log\cot s\right)^p - \lambda\right)_+ ds.$$

(iv) For any $(x,y) \in S$, $U_{\lambda,p}(x,y) \geq (|y|^p - \lambda)_+$.

Proof. (i) This is an immediate consequence of the following property of $\mathcal{U}_{\lambda,p}$: for all $\alpha \in \mathbb{R}$ and $\beta > 0$,

$$\mathcal{U}_{\lambda,p}(\alpha,\beta) = \mathcal{U}_{\lambda,p}(-\alpha,\beta) = \mathcal{U}_{\lambda,p}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right).$$

This can be established by substituting $t := -t$ and $t := 1/t$ in (2.1).

(ii) It suffices to prove the first estimate, then the second follows immediately from the harmonicity of $U_{\lambda,p}$. We have, after the substitution $t = s\exp(\pi y/2)$,

$$U_{\lambda,p}(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\frac{\pi}{2}x)\left(\left|\frac{2}{\pi}\log|s| + y\right|^p - \lambda\right)_+}{\left(s - \sin(\frac{\pi}{2}x)\right)^2 + \cos^2(\frac{\pi}{2}x)} \mathrm{d}s. \tag{2.3}$$

Since for any $s \in \mathbb{R}$ the function $y \mapsto \left(\left|\frac{2}{\pi}\log|s| + y\right|^p - \lambda\right)_+$ is convex, the claim follows.

(iii) Since $\partial_{xx}^2 U_{\lambda,p} \leq 0$, (i) implies that $x \mapsto U_{\lambda,p}(x,0)$ is nonincreasing on $[0,1]$. Consequently, if $x \in [-1,1]$, then (2.3) gives

$$\begin{aligned}
U_{\lambda,p}(x,0) \leq U_{\lambda,p}(0,0) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left(\left|\frac{2}{\pi}\log|s|\right|^p - \lambda\right)_+}{s^2+1}\,\mathrm{d}s \\
&= \frac{4}{\pi}\int_1^{\infty} \frac{\left((\frac{2}{\pi}\log s)^p - \lambda\right)_+}{s^2+1}\,\mathrm{d}s \\
&= \frac{4}{\pi}\int_0^{\pi/4}\left(\left(\frac{2}{\pi}\log\cot s\right)^p - \lambda\right)_+ \mathrm{d}s,
\end{aligned}$$

as desired.

(iv) Fix $y \in \mathbb{R}$ and apply Jensen's inequality in (2.3), with respect to the convex function $t \mapsto (|t+y|^p - \lambda)_+$. We get

$$U_{\lambda,p}(x,y) \geq \left(\left|\int_{-\infty}^{\infty} \frac{2}{\pi}\log|s| \cdot \frac{1}{\pi}\frac{\cos(\frac{\pi}{2}x)}{\left(s - \sin(\frac{\pi}{2}x)\right)^2 + \cos^2(\frac{\pi}{2}x)}\mathrm{d}s + y\right|^p - \lambda\right)_+,$$

which is the claim, since the integral inside is equal to 0 (simply substitute $s := 1/s$). ☐

3. Proofs of (1.2), (1.3), (1.4) and (1.5)

Our reasoning will depend heavily on the theory of continuous-time martingales; let us briefly introduce the necessary notions. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let X, Y be two real adapted càdlàg martingales, i.e., with right-continuous trajectories that have limits from the left. The symbols $[X,X]$ and $[Y,Y]$ will stand for the square brackets of X and Y, respectively; see e.g. Dellacherie and Meyer [13] for the definition. Following Bañuelos and Wang [2] and Wang [27], we say that Y is *differentially subordinate* to X, if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t. Furthermore, X and Y are said to be *orthogonal*, if their bracket $[X, Y]$ (defined by the polarization formula $[X, Y] = ([X+Y, X+Y] - [X-Y, X-Y])/4$) is constant.

In our further considerations, the following fact from stochastic analysis will be of importance. Recall that for any real martingale X there exists a unique continuous local martingale part X^c of X satisfying

$$[X, X]_t = |X_0|^2 + [X^c, X^c]_t + \sum_{0<s\leq t} |\triangle X_s|^2$$

for all $t \geq 0$. Here $\triangle X_s = X_s - X_{s-}$ denotes the jump of X at time s. Furthermore, we have that $[X^c, X^c] = [X, X]^c$, the pathwise continuous part of $[X, X]$. We will require the following statement, which appears as Lemma 2.1 in [3].

Lemma 3.1. *If X and Y are semimartingales, then Y is differentially subordinate and orthogonal to X if and only if Y^c is differentially subordinate and orthogonal to X^c, $|Y_0| \leq |X_0|$ and Y has continuous paths.*

Now we are ready to establish the following auxiliary estimate.

Lemma 3.2. *Suppose that X and Y are orthogonal martingales satisfying the conditions $||X||_\infty \leq 1$, $Y_0 \equiv 0$ and such that Y is differentially subordinate to X. Then for any $\lambda \geq 0$ and $p \geq 1$,*

$$\sup_{t\geq 0} \mathbb{E}(|Y_t|^p - \lambda)_+ \leq \frac{4}{\pi} \int_0^{\pi/4} \left(\left(\frac{2}{\pi} \log \cot s \right)^p - \lambda \right)_+ ds. \tag{3.1}$$

The inequality is sharp.

Proof. Let $t \geq 0$ be fixed. Since $U_{\lambda,p}$ is of class C^∞ in the interior of the strip S, we may apply Itô's formula to obtain

$$U_{\lambda,p}(X_t, Y_t) = I_0 + I_1 + I_2 + \frac{1}{2} I_3 + I_4,$$

where

$$\begin{aligned}
I_0 &= U_{\lambda,p}(X_0, Y_0), \\
I_1 &= \int_{0+}^t \partial_x U_{\lambda,p}(X_{s-}, Y_s) \mathrm{d}X_s + \int_{0+}^t \partial_y U_{\lambda,p}(X_{s-}, Y_s) \mathrm{d}Y_s, \\
I_2 &= \int_{0+}^t \partial^2_{xy} U_{\lambda,p}(X_{s-}, Y_s) \mathrm{d}[X^c, Y]_s, \\
I_3 &= \int_{0+}^t \partial^2_{xy} U_{\lambda,p}(X_{s-}, Y_s) \mathrm{d}[X^c, X^c]_s + \int_{0+}^t \partial^2_{yy} U_{\lambda,p}(X_{s-}, Y_s) \mathrm{d}[Y, Y]_s, \\
I_4 &= \sum_{0<s\leq t} \big[U_{\lambda,p}(X_s, Y_s) - U_{\lambda,p}(X_{s-}, Y_s) - \partial_x U_{\lambda,p}(X_{s-}, Y_s) \Delta X_s \big].
\end{aligned} \tag{3.2}$$

Note that we have used above the equalities $Y_{s-} = Y_s$ and $Y = Y^c$, which are due to the continuity of paths of Y. Let us analyse the terms I_1 through I_4 separately: here we will combine Lemma 3.1 with the properties of $U_{\lambda,p}$ studied in the previous section. By Lemma 2.1 (iii) we have

$$I_0 \leq \frac{4}{\pi} \int_0^{\pi/4} \left(\left(\frac{2}{\pi} \log \cot s \right)^p - \lambda \right)_+ \mathrm{d}s.$$

The term I_1 has zero expectation, by the properties of the stochastic integrals. We have $I_2 = 0$ in view of the orthogonality of X and Y. The differential subordination together with Lemma 2.1 (ii) imply

$$I_3 \leq \int_0^t \partial^2_{xx} U_{\lambda,p}(X_s, Y_s) d[X^c, X^c]_s + \int_0^t \partial^2_{yy} U_{\lambda,p}(X_s, Y_s) d[X^c, X^c]_s = 0.$$

Finally, we have that $I_4 \leq 0$, by the concavity of $U_{\lambda,p}(\cdot, y)$ for any fixed $y \in \mathbb{R}$: see Lemma 2.1 (ii). Therefore, by the last part of that lemma,

$$\mathbb{E}(|Y_t|^p - \lambda)_+ \leq \mathbb{E}U_{\lambda,p}(X_t, Y_t) \leq \frac{4}{\pi} \int_0^{\pi/4} \left(\left(\frac{2}{\pi} \log \cot s \right)^p - \lambda \right)_+ \mathrm{d}s \tag{3.3}$$

and it suffices to take supremum over t to obtain (3.1). The optimality of the constant on the right will follow from the sharpness of (1.2): see the end of Section 4 below. □

The next step in the analysis is to establish the following statement.

Lemma 3.3. *Suppose that X and Y are orthogonal martingales satisfying the conditions $||X||_\infty \leq 1$, $Y_0 \equiv 0$ and such that Y is differentially subordinate to X. Then for any p and any event A,*

$$\sup_{t \geq 0} \mathbb{E}|Y_t|^p 1_A \leq \frac{2^{p+2}}{\pi^{p+1}} \int_0^{\pi \mathbb{P}(A)/4} (\log \cot s)^p \, ds$$

and the number on the right cannot be replaced by a smaller one.

Proof. Fix t, an event A and let

$$\lambda = \left(\frac{2}{\pi} \log \cot \frac{\pi \mathbb{P}(A)}{4} \right)^p \geq 0.$$

Next, consider the splitting $A = A^+ \cup A^-$, where

$$A^+ = A \cap \{|Y_t|^p \geq \lambda\} \quad \text{and} \quad A^- = A \cap \{|Y_t|^p < \lambda\}.$$

By the previous lemma, we have

$$\begin{aligned}
\mathbb{E}(|Y_t|^p - \lambda)1_{A^+} \leq \mathbb{E}(|Y_t|^p - \lambda)_+ &\leq \frac{4}{\pi} \int_0^{\pi/4} \left(\left(\frac{2}{\pi} \log \cot s \right)^p - \lambda \right)_+ \mathrm{d}s \\
&= \frac{4}{\pi} \int_0^{\pi \mathbb{P}(A)/4} \left(\left(\frac{2}{\pi} \log \cot s \right)^p - \lambda \right) \mathrm{d}s \\
&= \frac{2^{p+2}}{\pi^{p+1}} \int_0^{\pi \mathbb{P}(A)/4} (\log \cot s)^p \, \mathrm{d}s - \lambda \mathbb{P}(A).
\end{aligned}$$

Furthermore, we obviously have $\mathbb{E}(|Y_t|^p - \lambda)1_{A^-} \leq 0$. Adding the latter two estimates gives

$$\mathbb{E}(|Y_t|^p - \lambda)1_A \leq \frac{2^{p+2}}{\pi^{p+1}} \int_0^{\pi \mathbb{P}(A)/4} (\log \cot s)^p \, \mathrm{d}s - \lambda \mathbb{P}(A)$$

or, equivalently,

$$\mathbb{E}|Y_t|^p 1_A \leq \frac{2^{p+2}}{\pi^{p+1}} \int_0^{\pi \mathbb{P}(A)/4} (\log \cot s)^p \, \mathrm{d}s.$$

It remains to take the supremum over $t \geq 0$ to get the desired estimate. Its sharpness is deferred to the end of Section 4. □

The next result can be regarded as a probabilistic version of (1.2) and is of independent interest.

Theorem 3.4. *Suppose that $X = (X^1, X^2, \ldots, X^N)$ is a bounded martingale taking values in ℓ_∞^N and let $Y = (Y^1, Y^2, \ldots, Y^N)$ be an ℓ_∞^N-valued martingale such that for each j, Y^j and X^j are orthogonal and Y^j is differentially subordinate to X^j. Then we have the sharp inequality*

$$\sup_{t\geq 0} \big|\big| ||Y_t||_{\ell_\infty^N} \big|\big|_p \leq N^{1/p} \cdot \left(\frac{2^{p+2}}{\pi^{p+1}} \int_0^{\pi/(4N)} (\log \cot s)^p \, ds \right)^{1/p} \sup_{t\geq 0} \big|\big| ||X_t||_{\ell_\infty^N} \big|\big|_\infty .$$

Proof. By homogeneity, we may assume that $||X||_{\ell_\infty^N} \leq 1$; then each coordinate X^j is a martingale taking values in the interval $[-1, 1]$. Suppose that $t \geq 0$ is fixed. Then there are pairwise disjoint events A_1, A_2, ..., A_N such that

$$||Y_t||_{\ell_\infty^N}^p = \sum_{j=1}^N |Y_t^j|^p 1_{A_j}.$$

and hence, by the previous lemma,

$$\mathbb{E}||Y_t||_{\ell_\infty^N}^p = \sum_{j=1}^N \mathbb{E}|Y_t^j|^p 1_{A_j} \leq \sum_{j=1}^N \frac{2^{p+2}}{\pi^{p+1}} \int_0^{\pi\mathbb{P}(A_j)/4} (\log \cot s)^p \, \mathrm{d}s.$$

However, the function $u \mapsto \int_0^u (\log \cot s)^p \, \mathrm{d}s$ is concave: indeed, the integrand is nonincreasing. Consequently, by Jensen's inequality, we obtain

$$\mathbb{E}||Y_t||_{\ell_\infty^N}^p \leq N \cdot \frac{2^{p+2}}{\pi^{p+1}} \int_0^{\pi/(4N)} (\log \cot s)^p \, \mathrm{d}s.$$

It remains to take the supremum over t to get the claim. The sharpness will be made clear at the end of Section 4. □

Now let us deduce our main result for the Hilbert transform.

Proof of (1.2). This follows from a well-known and straightforward argument. Let $B = (B_t)_{t\geq 0}$ be a planar Brownian motion starting from 0 and let τ denote the first moment B hits the unit circle. Let f be a fixed function on the unit circle $\mathbb{T}$, taking values in the unit ball of ℓ_∞^N. Denote by u and v the harmonic extensions of f and $\mathcal{H}f$ to the unit disc. Then the processes X, Y given by $X_t = u(B_{\tau\wedge t})$, $Y_t = v(B_{\tau\wedge t})$ (for $t \geq 0$) are orthogonal martingales and Y is differentially subordinate to X (more precisely, these properties hold for each pair X^j and Y^j of coordinates). This follows at once from the identities

$$\begin{aligned}
[X^j, X^j]_t &= u^2(\xi) + \int_0^{\tau\wedge t} |\nabla u^j(B_s)|^2 \mathrm{d}s, \\
[Y^j, Y^j]_t &= \int_0^{\tau\wedge t} |\nabla v^j(B_s)|^2 \mathrm{d}s = \int_0^{\tau\wedge t} |\nabla u^j(B_s)|^2 \mathrm{d}s, \\
[X^j, Y^j]_t &= \int_0^{\tau\wedge t} \nabla u^j(B_s) \cdot \nabla v^j(B_s) \mathrm{d}s = 0,
\end{aligned}$$

where the latter equality is due to Cauchy-Riemann equations. But u takes values in the unit ball of ℓ_∞^N, since so does f. Consequently, the martingale X also has this boundedness property. Therefore, by (1.2), we may write

$$\begin{aligned}||\mathcal{H}f||_{L^p(\mathbb{T};\ell_\infty^N)} &\leq \sup_{t\geq 0}\big|\big|\,||Y_t||_{\ell_\infty^N}\big|\big|_p\\ &\leq N^{1/p}\cdot\left(\frac{2^{p+2}}{\pi^{p+1}}\int_0^{\pi/(4N)}(\log\cot s)^p\,\mathrm{d}s\right)^{1/p}\sup_{t\geq 0}\big|\big|\,||X_t||_{\ell_\infty^N}\big|\big|_\infty\\ &\leq N^{1/p}\cdot\left(\frac{2^{p+2}}{\pi^{p+1}}\int_0^{\pi/(4N)}(\log\cot s)^p\,\mathrm{d}s\right)^{1/p}||f||_{L^\infty(\mathbb{T};\ell_\infty^N)},\end{aligned}$$

which gives the first estimate. To obtain (1.3), we use duality argument: clearly, we have

$$||\mathcal{H}f||_{L^1(\mathbb{T};\ell_1^N)}=\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{j=1}^N|\mathcal{H}f^j|\mathrm{d}\theta=\sup\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{j=1}^N\mathcal{H}f^jg^j\,\mathrm{d}\theta,$$

where the supremum is taken over all functions $g=(g^1,g^2,\ldots,g^N)$ on $\mathbb{T}$ satisfying $||g||_{L^\infty(\mathbb{T};\ell_\infty^N)}\leq 1$. Now, for any such g, we may write

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{j=1}^N\mathcal{H}f^jg^j\,\mathrm{d}\theta&=-\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{j=1}^N f^j\mathcal{H}g^j\,\mathrm{d}\theta\\ &\leq ||\mathcal{H}g||_{L^{p'}(\mathbb{T};\ell_\infty^N)}||f||_{L^p(\mathbb{T};\ell_1^N)}\\ &\leq\left(N\cdot\frac{2^{p'+2}}{\pi^{p'+1}}\int_0^{\pi/(4N)}(\log\cot s)^{p'}\,\mathrm{d}s\right)^{1/p'}||f||_{L^p(\mathbb{T};\ell_1^N)},\end{aligned}$$

where in the last passage we have applied (1.2) to g. This proves the claim. □

Finally, we are ready to establish the inequalities of Corollary 1.2.

Proof of (1.4) *and* (1.5). To show the exponential bound, observe that (1.2) implies, for any $n\geq 1$,

$$||\mathcal{H}f||^n_{L^n(\mathbb{T};\ell_\infty^N)}\leq\frac{4N}{\pi}\int_0^{\pi/(4N)}\left(\frac{2}{\pi}\log\cot s\right)^n\mathrm{d}s.$$

Divide both sides by $n!$, sum over all n add 1 to both sides to get

$$\frac{1}{2\pi}\int_{\mathbb{T}}\exp\left(K||\mathcal{H}f(e^{i\theta})||_{\ell_\infty^N}\right)\mathrm{d}\theta\leq\frac{4N}{\pi}\int_0^{\pi/(4N)}\exp\left(\frac{2K}{\pi}\log\cot s\right)\mathrm{d}s,$$

which is (1.4) (use the substitution $u=\cot s$). To show the LlogL estimate, we use duality, as in the above proof of (1.3). We will need an auxiliary bound

$$xy\leq\Psi(x)+e^y \tag{3.4}$$

valid for all nonnegative x, y (we leave the straightforward proof to the reader). We have

$$\begin{aligned}||\mathcal{H}f||_{L^1(\mathbb{T};\ell_1^N)}&=\sup\left\{-\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{j=1}^N f^j(e^{i\theta})\mathcal{H}g^j(e^{i\theta})\mathrm{d}\theta\,:\,||g||_{L^\infty(\mathbb{T};\ell_\infty^N)}\leq 1\right\}\\ &\leq\sup\left\{\frac{L}{2\pi}\int_{\mathbb{T}}||f(e^{i\theta})||_{\ell_1^N}\left|\left|\frac{\mathcal{H}g(e^{i\theta})}{L}\right|\right|_{\ell_\infty^N}\mathrm{d}\theta:||g||_{L^\infty(\mathbb{T};\ell_\infty^N)}\leq 1\right\}\end{aligned}$$

and, by (3.4) and (1.4) (applied to $K = 1/L$),

$$\begin{aligned}
&\frac{L}{2\pi}\int_{\mathbb{T}} ||f(e^{i\theta})||_{\ell_1^N} \left|\left|\frac{\mathcal{H}g(e^{i\theta})}{L}\right|\right|_{\ell_\infty^N} \mathrm{d}\theta \\
&\leq \frac{L}{2\pi}\int_{\mathbb{T}} \Psi\left(||f(e^{i\theta})||_{\ell_1^N}\right)\mathrm{d}\theta + \frac{L}{2\pi}\int_{\mathbb{T}} \exp\left(\left|\left|\frac{\mathcal{H}g(e^{i\theta})}{L}\right|\right|_{\ell_\infty^N}\right)\mathrm{d}\theta \\
&\leq \frac{L}{2\pi}\int_{\mathbb{T}} \Psi\left(||f(e^{i\theta})||_{\ell_1^N}\right)\mathrm{d}\theta + \frac{4LN}{\pi}\int_{\cot(\pi/(4N))}^{\infty} \frac{u^{2/(\pi L)}}{u^2+1}\mathrm{d}u.
\end{aligned}$$

This completes the proof. □

4. Sharpness

In this section, we show that the constants appearing in (1.2), (1.3), (1.4), (1.5) as well as in the estimates of Lemmas 3.2, 3.3 and Theorem 3.4 cannot be improved. First let us focus on the estimate (1.2). Note that the mapping

$$G(z) = (-2i/\pi)\log[(iz-1)/(z-i)] - 1$$

is conformal, satisfies $G(0) = 0$ and sends the unit disc onto the strip $[-1,1]\times\mathbb{R}$. We easily derive that $w = \operatorname{Re} G$ and its conjugate $\mathcal{H}w = \operatorname{Im} G$ admit the formulas

$$w(e^{i\theta}) = 1_{\{|\theta|\leq\pi/2\}} - 1_{\{|\theta|>\pi/2\}}, \qquad \mathcal{H}w(e^{i\theta}) = -\frac{2}{\pi}\log\left|\frac{1+\sin\theta}{\cos\theta}\right|$$

for $\theta\in[-\pi,\pi]$. Now, given a positive integer N, consider the set

$$A = \Big\{\theta\in[-\pi,\pi] \ : \ |\theta-\pi/2| < \pi/(2N) \quad \text{or} \quad |\theta+\pi/2| < \pi/(2N)\Big\},$$

which is a sum of two intervals of length π/N, with centers at $-\pi/2$ and $\pi/2$. The function $\mathcal{H}w$ satisfies the symmetry condition

$$|\mathcal{H}w(e^{i\theta}) = |\mathcal{H}w(e^{i(\pi-\theta)})| = |\mathcal{H}w(e^{-i\theta})|$$

for all $\theta\in[-\pi,\pi]$, and therefore

$$\begin{aligned}
\frac{1}{2\pi}\int_A |\mathcal{H}w(e^{i\theta})|^p\mathrm{d}\theta &= \frac{2}{\pi}\int_{\pi/2}^{\pi/2+\pi/(2N)} |\mathcal{H}w(e^{i\theta})|^p\mathrm{d}\theta \\
&= \frac{2^{p+1}}{\pi^{p+1}}\int_{\pi/2}^{\pi/2+\pi/(2N)} \left|\log\left|\frac{1+\sin\theta}{\cos\theta}\right|\right|^p \mathrm{d}\theta \\
&= \frac{2^{p+1}}{\pi^{p+1}}\int_0^{\pi/(2N)} \left|\log\left|\frac{1+\cos\theta}{\sin\theta}\right|\right|^p \mathrm{d}\theta \\
&= \frac{2^{p+2}}{\pi^{p+1}}\int_0^{\pi/(4N)} (\log\cot s)^p\mathrm{d}s.
\end{aligned}$$

Now consider the pairwise disjoint subsets $A_1 = A$, $A_2 = A+\pi/N$, $A_3 = A+2\pi/N$, ..., $A_N = A+(N-1)\pi/N$ of $[-\pi,\pi]$ (here, as usual, $A+u = \{x+u : x\in A\}$ and we identify x and y if their difference is a multiple of 2π). Composing the function G with an appropriate rotation $z\mapsto e^{i\varphi}z$, we see that for any $j\in\{1,2,\ldots,N\}$, there is a function $w^j:\mathbb{T}\to\{-1,1\}$, such that

$$\frac{1}{2\pi}\int_{A_j} |\mathcal{H}w^j(e^{i\theta})|^p\mathrm{d}\theta = \frac{2^{p+2}}{\pi^{p+1}}\int_0^{\pi/(4N)} (\log\cot s)^p\mathrm{d}s.$$

Therefore, if we take $w=(w^1,w^2,\ldots,w^N)$, we see that $||w||_{L^\infty(\mathbb{T};\ell_\infty^N)}=1$ and

$$\begin{aligned}||\mathcal{H}w||_{L^p(\ell_\infty^N)}&\geq\left(\frac{1}{2\pi}\sum_{j=1}^N\int_{A_j}|\mathcal{H}w^j|^p\mathrm{d}\theta\right)^{1/p}\\&=\left(N\cdot\frac{2^{p+2}}{\pi^{p+1}}\int_0^{\pi/(4N)}(\log\cot s)^p\mathrm{d}s\right)^{1/p}.\end{aligned}$$

Therefore, the inequality (1.2) is sharp. This also proves that the constants involved in the estimates of Lemmas 3.2, 3.3 and Theorem 3.4 are also the best possible: indeed, if any of them could be improved, then it would be possible to decrease the constant in (1.2). Finally, since the above extremal example is the same for each p, we immediately obtain the sharpness of the exponential bound (1.4).

It remains to show that the constants in (1.3) and (1.5) are the best as well. This can be deduced from the sharpness of (1.2) and (1.4) with the use of duality argument. We will only present the details for the first estimate, and leave the analysis of (1.5) to the reader. So, assume that for a given $1<p\leq\infty$, the optimal constant in (1.3) equals β_p. Fix $f\in L^\infty(\mathbb{T};\ell_\infty^N)$ and observe that

$$||\mathcal{H}f||_{L^{p'}(\mathbb{T};\ell_\infty^N)}=\sup\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{j=1}^N\mathcal{H}f^jg^j\mathrm{d}\theta,$$

where the supremum is taken over all $g=(g^1,g^2,\ldots,g^N)$ on $\mathbb{T}$ satisfying $||g||_{L^p(\mathbb{T};\ell_1^N)}\leq 1$. Now, for any such g we may write

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{j=1}^N\mathcal{H}f^jg^j\mathrm{d}\theta&=\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{j=1}^Nf^j\mathcal{H}g^j\mathrm{d}\theta\\&\leq||f||_{L^\infty(\mathbb{T};\ell_\infty^N)}||\mathcal{H}g||_{L^1(\mathbb{T};\ell_1^N)}\leq\beta_p||f||_{L^\infty(\mathbb{T};\ell_\infty^N)}.\end{aligned}$$

Here we have used the assumed fact that (1.3) holds with the constant β_p. Consequently, $||\mathcal{H}f||_{L^{p'}(\mathbb{T};\ell_\infty^N)}\leq\beta_p||f||_{L^\infty(\mathbb{T};\ell_\infty^N)}$ and by the sharpness of (1.2), we get

$$\beta_p\geq\left(N\cdot\frac{2^{p'+2}}{\pi^{p'+1}}\int_0^{\pi/(4N)}(\log\cot s)^{p'}\,\mathrm{d}s\right)^{1/p'}.$$

This is precisely the claim.

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