ON THE ACTION OF THE HILBERT TRANSFORM ON ℓ_1^N-VALUED FUNCTIONS

ADAM OSĘKOWSKI

Abstract. Let $\mathbb{H}$ be a separable Hilbert space. The periodic Hilbert transform $\mathcal{H}$ is bounded as an operator on $L_p(\mathbb{T};\ell_1^N(\mathbb{H}))$, $1<p<\infty$, since $\ell_1^N(\mathbb{H})$ is a UMD space. We prove that there is a finite constant c_p depending only on p such that

$$c_p^{-1}(\ln N+1)\leq ||\mathcal{H}||_{L_p(\mathbb{T};\ell_1^N(\mathbb{H}))\to L_p(\mathbb{T};\ell_1^N(\mathbb{H}))}\leq c_p(\ln N+1).$$

The proof uses probabilistic methods and exploits bounds for differentially subordinate martingales.

1. Introduction

Let $\mathbb{B}$ be a real or complex Banach space and let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing family of sub-σ-algebras of $\mathcal{F}$. The space $\mathbb{B}$ is called a UMD-space (*unconditional for martingale differences*) if for some $p\in(1,\infty)$ (equivalently, for all $1<p<\infty$) there is a finite constant β_p such that the following holds. If $(d_n)_{n\geq 0}$ is a $\mathbb{B}$-valued martingale difference sequence and $(\varepsilon_n)_{n\geq 0}$ is an arbitrary deterministic sequence of signs, then

$$\left\|\sum_{k=0}^n \varepsilon_k d_k\right\|_{L_p(\Omega;\mathbb{B})}\leq \beta_p\left\|\sum_{k=0}^n d_k\right\|_{L_p(\Omega;\mathbb{B})}.$$

In this definition, the filtration must vary and so must the probability space, unless it is assumed to be nonatomic. Let $UMD_p(\mathbb{B})$ denote the smallest value of β_p allowed in the above estimate.

UMD spaces form right environment into which many classical results from harmonic analysis on Hilbert spaces can be carried over. One of the motivations for the study of these spaces came from an attempt to extend the work of M. Riesz on the L_p-boundedness of the Hilbert transform, and that of Calderón and Zygmund on more general singular integral operators, to the case of functions with values in a Banach space. Let us be more specific. Suppose that $\mathbb{T}\simeq(-\pi,\pi]$ denotes the unit circle on the complex plane. Then $\mathcal{H}$, the periodic Hilbert transform, is an operator acting on functions $f\in L_1(\mathbb{T};\mathbb{R})$ by the formula

$$\mathcal{H}f(e^{i\theta})=\frac{1}{2\pi}\text{p.v.}\int_{-\pi}^{\pi} f(e^{it})\cot\frac{\theta-t}{2}\mathrm{d}t. \tag{1.1}$$

In the twenties, M. Riesz proved that for any $1<p<\infty$, the periodic Hilbert transform is bounded as an operator on $L_p(\mathbb{T};\mathbb{R})$; this automatically leads to the analogous statement for the non-periodic version of the Hilbert transform. This fact was further generalized by Calderón and Zygmund to a much wider setting of singular integral operators associated with odd kernels.

There is a natural question concerning the version of the above results for Banach-space valued functions. It was soon realized that not all spaces are well-behaved, even for the Hilbert transform: Bochner and Taylor [2] showed that $||\mathcal{H}||_{L_p(\mathbb{T};\ell_1)\to L_p(\mathbb{T};\ell_1)}=\infty$. It turns out that the periodic Hilbert

2010 *Mathematics Subject Classification.* Primary: 60G42. Secondary: 60G44.

Key words and phrases. Martingale, analytic, transform, UMD space, best constants, plurisubharmonic function.

transform is bounded as an operator on $L_p(\mathbb{T};\mathbb{B})$ if and only if $\mathbb{B}$ has the UMD property. This equivalence is due to Burkholder and McConnell [4], who showed that UMD spaces are well-behaved for the Hilbert transform, and Bourgain [3], who established the reverse implication. This, by the use of Calderón-Zygmund method of rotations, showed that UMD spaces form a natural context for the study of singular integrals with odd kernels. What about the specific dependence between the norm $||\mathcal{H}||_{L_p(\mathbb{T};\mathbb{B})\to L_p(\mathbb{T};\mathbb{B})}$ and the UMD constants $UMD_p(\mathbb{B})$? It follows from [3] and [4] that there is an absolute constant C such that

$$C^{-1}(UMD_p(\mathbb{B}))^{1/2}\leq ||\mathcal{H}||_{L_p(\mathbb{T};\mathbb{B})\to L_p(\mathbb{T};\mathbb{B})}\leq C(UMD_p(\mathbb{B}))^2. \tag{1.2}$$

There is a very interesting question whether this quadratic equivalence can be improved to the linear dependence. This problem has gained a lot of interest in the literature, and despite many attempts and partial results, seems to be completely open at the moment. The principal goal of this paper is to study this subject in the context of the spaces $\ell_1^N(\mathbb{H})$, where $\mathbb{H}$ is a separable Hilbert space. It is well known that $UMD_p(\ell_1^N(\mathbb{H}))\simeq \ln N+1$ (cf. [9]): for each p there is a constant c_p depending only on p such that for each N,

$$c_p^{-1}(\ln N+1)\leq UMD_p(\ell_1^N(\mathbb{H}))\leq c_p(\ln N+1)$$

and hence, by (1.2), we have the two-sided bound

$$C^{-1}c_p^{-1}(\ln N+1)^{1/2}\leq ||\mathcal{H}||_{L_p(\mathbb{T};\ell_1^N(\mathbb{H}))\to L_p(\mathbb{T};\ell_1^N(\mathbb{H}))}\leq Cc_p(\ln N+1)^2.$$

We will improve this result to the linear dependence on both sides.

Theorem 1.1. *For any $1<p<\infty$ and any N we have the estimates*

$$\pi^{-1}(\ln N+1)\leq ||\mathcal{H}||_{L_1(\mathbb{T};\ell_1^N(\mathbb{H}))\to L_{1,\infty}(\mathbb{T};\ell_1^N(\mathbb{H}))}\leq 4(\ln N+1) \tag{1.3}$$

and

$$(2\pi)^{-1}(\ln N+1)\leq ||\mathcal{H}||_{L_p(\mathbb{T};\ell_1^N(\mathbb{H}))\to L_p(\mathbb{T};\ell_1^N(\mathbb{H}))}\leq \frac{288p^2}{p-1}(\ln N+1). \tag{1.4}$$

The proof will depend heavily on stochastic analysis and, in particular, on properties of continuous-time martingales. Our argumentation will be split into two sections. Section 2 is devoted to the upper bounds in (1.3) and (1.4). We will first establish certain probabilistic versions of these inequalities for $\ell_1^N(\mathbb{H})$-valued martingales. Our main tool is the so-called Burkholder's method: using a certain special function, enjoying appropriate size requirements and concavity, we will prove the martingale weak-type bound and then deduce the strong-type estimate with the use of standard extrapolation (good-lambda inequalities). Section 3 is devoted to the lower bounds in (1.3) and (1.4). A natural idea is to construct appropriate examples: we follow this path but we again need to incorporate certain probabilistic arguments. Namely, we show that the validity of a strong-type estimate for Hilbert transform acting on $\ell_1^N(\mathbb{C})$-valued functions implies the same weak- and strong-type inequalities for *analytic* martingales with values in $\ell_1^N(\mathbb{C})$. We conclude by presenting an inductive efficient construction of a certain analytic martingale, thus providing the desired lower bounds for the constants.

2. An upper bound in (1.3) and (1.4)

2.1. **Background and notation.** We start with recalling some basic facts from stochastic analysis which will be needed in our argumentation. Suppose that $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space, equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Suppose that X, Y are adapted continuous-path martingales taking values in a certain fixed separable Hilbert space $\mathbb{H}$ with the norm $|\cdot|$ and the scalar product $\langle\cdot,\cdot\rangle$ (with no loss of generality, we may assume that $\mathbb{H}=\ell^2$). The $X^*=\sup_{t\geq 0}|X_t|$, $Y^*=\sup_{t\geq 0}|Y_t|$ denote the maximal functions of X and

Y, respectively. Let $[X]$, $[Y]$ denote the quadratic variation processes (square brackets) of X and Y; see Dellacherie and Meyer [7] for details when $\mathbb{H}=\mathbb{R}$, and extend the definition to the higher-dimensional setting by $[X]=\sum_{n=1}^\infty [X^n]$, where X^n denotes the n-th coordinate of X. Following Bañuelos and Wang [1] and Wang [10], the process Y is differentially subordinate to X, if, with probability 1, the difference $[X]-[Y]=([X]_t-[Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t. For example, this notion arises in the context of stochastic integration. Suppose that X is a martingale, H is a predictable process and let $Y=H\cdot X$ be the stochastic integral of H with respect to X. If H takes values in $[-1,1]$, then Y is differentially subordinate to X; this follows at once from

$$[X]_t-[Y]_t=\int_0^t (1-H_s^2)\mathrm{d}[X]_s, \qquad t\geq 0.$$

In what follows, we will also exploit the properties of the so-called *analytic* martingales. Recall that for continuous-path continuous-time martingales X, Y taking values in $\mathbb{R}^N$, the $\mathbb{C}^N$-valued martingale $Z=X+iY$ is called analytic, if for any j, $k\in\{1,2,\ldots,N\}$ we have $[Z^j,Z^k]=0$, or, equivalently, $[X^j,X^k]=[Y^j,Y^k]$ and $[X^j,Y^k]=-[Y^j,X^k]$, where, as usual, X^k denotes the k-th coordinate of X.

Sometimes we will also need to work with martingales taking values in $\ell_1^N(\mathbb{H})$, and we have decided to use the same letters X, Y in such a case. This should not lead to any confusion: the target space will always be specifically given in the formulation of each statement.

2.2. **An inequality for Hilbert-space-valued martingales.** Let $N\geq 1$ be a fixed integer. A crucial object for our further considerations is the special function $u:\mathbb{H}\times\mathbb{H}\to\mathbb{R}$, which is given by

$$u(x,y)=\begin{cases} N(|y|^2-|x|^2)/2 & \text{if } |x|+|y|\leq N^{-1},\\ |y|-|x|\ln\big(N(|x|+|y|)\big)-(2N)^{-1} & \text{if } N^{-1}<|x|+|y|\leq 1,\\ 1-(2N)^{-1}-(\ln N+1)|x| & \text{if } |x|+|y|>1.\end{cases}$$

We will need the following majorization property of u.

Lemma 2.1. *For any x, $y\in\mathbb{H}$ such that $|y|\leq 1$ we have*

$$(|y|-(2N)^{-1})_+-(\ln N+1)|x|\leq u(x,y)\leq 1-(2N)^{-1}-(\ln N+1)|x|. \tag{2.1}$$

Proof. Since the dependence of u on x and y is through the norms $|x|$ and $|y|$ only, we may assume that $\mathbb{H}=\mathbb{R}$ and x, $y\geq 0$. One easily checks that for any fixed $y\in[0,1]$, the function $x\mapsto u(x,y)$ is concave on $[0,\infty)$; hence, the function $\varphi_y(x):=u(x,y)+(\ln N+1)x$ also has this property. Now, we see that $\varphi_y(x)=1-(2N)^{-1}$ for sufficiently large x, and

$$\varphi_y(0)=\begin{cases} Ny^2/2 & \text{if } y<N^{-1},\\ y-(2N)^{-1} & \text{if } y\geq N^{-1}\end{cases}$$

is not smaller than $(y-(2N)^{-1})_+$: when $y\in((2N)^{-1},N^{-1})$, this is equivalent to $(y-N^{-1})^2\geq 0$; for other y's, this is trivial. So, we have proved that $(y-(2N)^{-1})_+\leq\varphi_y(x)\leq 1-(2N)^{-1}$, the claim. □

The function u behaves nicely when composed with the differentially subordinate martingales.

Lemma 2.2. *Suppose that X, Y are $\mathbb{H}$-valued martingales such that Y takes values in the unit ball of $\mathbb{H}$ and is differentially subordinate to X. Then for any $t\geq 0$ we have*

$$\mathbb{E}u(X_t,Y_t)\leq 0.$$

Proof. Fix $t\geq 0$ and consider the stopping time $\tau=\inf\{s\geq 0:|X_s|+|Y_s|=1\}$. First we show that

$$\mathbb{E}u(X_t,Y_t)\leq\mathbb{E}u(X_{\tau\wedge t},Y_{\tau\wedge t}) \tag{2.2}$$

or, equivalently, $\mathbb{E}u(X_t,Y_t)1_{\{\tau\leq t\}}\leq \mathbb{E}u(X_\tau,Y_\tau)1_{\{\tau\leq t\}}$. This is easy: using the right inequality in (2.1) and applying Doob's optional sampling theorem to the supermartingale $-|X|$, we get

$$\begin{aligned}\mathbb{E}u(X_t,Y_t)1_{\{\tau\leq t\}}&\leq \mathbb{E}\big[1-(2N)^{-1}-(\ln N+1)|X_t|\big]1_{\{\tau\leq t\}}\\&\leq \mathbb{E}\big[1-(2N)^{-1}-(\ln N+1)|X_\tau|\big]1_{\{\tau\leq t\}}=\mathbb{E}u(X_\tau,Y_\tau)1_{\{\tau\leq t\}}.\end{aligned}$$

This establishes (2.2) and hence it is enough to prove that $\mathbb{E}u(X_{\tau\wedge t},Y_{\tau\wedge t})\leq 0$. If $\tau=0$, then $|X_\tau|=|X_0|\geq(|X_0|+|Y_0|)/2\geq 1/2$ and $u(X_{\tau\wedge t},Y_{\tau\wedge t})=u(X_0,Y_0)=1-(2N)^{-1}-(\ln N+1)|X_0|\leq 0$. So, we will be done if we show that $\mathbb{E}u(X_{\tau\wedge t},Y_{\tau\wedge t})1_{\{\tau>0\}}\leq 0$. On the set $\{\tau>0\}$ we have $|X_0|+|Y_0|<1$ and hence, by the continuity of paths of X and Y, we see that $|X_{\tau\wedge t}|+|Y_{\tau\wedge t}|\leq 1$. Consequently,

$$\mathbb{E}u(X_{\tau\wedge t},Y_{\tau\wedge t})1_{\{\tau>0\}}=\mathbb{E}v(X_{\tau\wedge t}1_{\{\tau>0\}},Y_{\tau\wedge t}1_{\{\tau>0\}}),$$

where

$$v(x,y)=\begin{cases}N(|y|^2-|x|^2)/2 & \text{if } |x|+|y|\leq N^{-1},\\ |y|-|x|\ln\big(N(|x|+|y|)\big)-(2N)^{-1} & \text{if } |x|+|y|\geq N^{-1}.\end{cases}$$

Now, recall the function $w:\mathbb{H}\times\mathbb{H}\to\mathbb{R}$, invented by Burkholder [5]:

$$w(x,y)=\begin{cases}|y|^2-|x|^2 & \text{if } |x|+|y|\leq 1,\\ 1-2|x| & \text{if } |x|+|y|>1.\end{cases}$$

This function has the property that if F, G are any $\mathbb{H}$-valued martingales such that G is differentially subordinate to F, then $\mathbb{E}w(F_t,G_t)\leq 0$ (cf. Wang [10]). It is easy to check the identity

$$v(x,y)=\frac{1}{2}\int_{N^{-1}}^\infty w(x/r,y/r)\mathrm{d}r,$$

which, by Fubini's theorem, implies that for any bounded F, G as above, we have $\mathbb{E}v(F_t,G_t)\leq 0$. It suffices to apply this property to the martingales $F=(X_{\tau\wedge s}1_{\{\tau>0\}})_{s\geq 0}$, $G=(Y_{\tau\wedge s}1_{\{\tau>0\}})_{s\geq 0}$ (which clearly inherit the differential subordination from the processes X and Y). □

2.3. **Inequalities for $\ell_1^N(\mathbb{H})$-valued martingales and the Hilbert transform.** Equipped with the auxiliary estimate of Lemma 2.2, we turn our attention to the context of $\ell_1^N(\mathbb{H})$-valued martingales.

Lemma 2.3. *Suppose that X, Y are $\ell_1^N(\mathbb{H})$-valued martingales such that Y takes values in the unit ball of $\ell_1^N(\mathbb{H})$ and for each $j=1,2,\ldots,N$, the martingale Y^j is differentially subordinate to X^j. Then for any $t\geq 0$ we have*

$$\mathbb{P}(||Y_t||_{\ell_1^N(\mathbb{H})}\geq 1)\leq 4(\ln N+1)||X||_{L_1(\ell_1^N)}. \tag{2.3}$$

Proof. For any $j=1,2,\ldots,N$, the coordinate Y^j takes values in the unit ball of $\mathbb{H}$. Consequently,

$$\mathbb{E}u(X_t^j,Y_t^j)\leq 0,\qquad t\geq 0,$$

in the light of the preceding lemma. By the left inequality in (2.1), this implies

$$\mathbb{E}(|Y_t^j|-(2N)^{-1})_+\leq 2(\ln N+1)\mathbb{E}|X_t^j|\leq 2(\ln N+1)||X^j||_{L_1}.$$

Now, pick an arbitrary event $A\in\mathcal{F}$ and consider its splitting into the sets

$$A_-=A\cap\{|Y_t^j|<(2N)^{-1}\}\quad\text{and}\quad A_+=A\cap\{|Y_t^j|\geq(2N)^{-1}\}.$$

Obviously, we have $\int_{A_-} |Y_t^j| d\mathbb{P} \leq (2N)^{-1}\mathbb{P}(A_-)$. Furthermore,

$$\begin{aligned}\int_{A_+} |Y_t^j| d\mathbb{P} &= \int_{A_+} (|Y_t^j| - (2N)^{-1}) d\mathbb{P} + (2N)^{-1}\mathbb{P}(A_+)\\ &\leq \mathbb{E}(|Y_t^j| - (2N)^{-1})_+ + (2N)^{-1}\mathbb{P}(A_+) \leq 2(\ln N+1)||X^j||_{L_1} + (2N)^{-1}\mathbb{P}(A_+).\end{aligned}$$

Summing the two latter estimates, we obtain

$$\int_A |Y_t^j| d\mathbb{P} \leq 2(\ln N+1)||X^j||_{L_1} + (2N)^{-1}\mathbb{P}(A).$$

Now write this bound for $j = 1, 2, \ldots, N$ and add all these estimates to get

$$\int_A ||Y_t||_{\ell_1^N(\mathbb{H})} d\mathbb{P} \leq 2(\ln N+1)||X||_{L_1(\ell_1^N)} + \frac{1}{2}\mathbb{P}(A).$$

Now, set $A = \{||Y_t||_{\ell_1^N(\mathbb{H})} \geq 1\}$. Then $\mathbb{P}(A) \leq \int_A ||Y_t||_{\ell_1^N} d\mathbb{P}$ and the above estimate implies $\mathbb{P}(A) \leq 4(\ln N+1)||X||_{L_1(\ell_1^N)}$. This is exactly what we have claimed in the statement of the lemma. □

We will need the following maximal version of the above bound.

Theorem 2.4. *Suppose that X, Y are $\ell_1^N(\mathbb{H})$-valued martingales such that for each $j = 1, 2, \ldots, N$, the martingale Y^j is differentially subordinate to X^j. Then*

$$\mathbb{P}(Y^* \geq 1) \leq 4(\ln N+1)||X||_{L_1(\ell_1^N)}. \tag{2.4}$$

Proof. If $||Y_0||_{\ell_1^N(\mathbb{H})} \geq 1$, then $||X_0||_{\ell_1^N(\mathbb{H})} \geq 1$, by the differential subordination, and hence

$$\mathbb{P}(||Y_0||_{\ell_1^N(\mathbb{H})} \geq 1) \leq \mathbb{E}||X_0||_{\ell_1^N(\mathbb{H})} 1_{\{||Y_0||_{\ell_1^N(\mathbb{H})} \geq 1\}}.$$

To control the size of the set $\{Y^* \geq 1\} \cap \{|Y_0| < 1\}$, consider the martingales $\tilde{X} = (X_t 1_{\{||Y_0||_{\ell_1^N(\mathbb{H})} < 1\}})_{t \geq 0}$ and $\tilde{Y} = (Y_t 1_{\{||Y_0||_{\ell_1^N(\mathbb{H})} < 1\}})_{t \geq 0}$. Fix $\varepsilon > 0$ and introduce the stopping time $\tau = \inf\{t : ||\tilde{Y}_t||_{\ell_1^N(\mathbb{H})} \geq 1-\varepsilon\}$. Clearly, we have

$$\{\tilde{Y}^* \geq 1\} \subseteq \{||\tilde{Y}_{\tau\wedge t}||_{\ell_1^N(\mathbb{H})} \geq 1-\varepsilon \text{ for some } t\} = \bigcup_{t \geq 0} \{||\tilde{Y}_{\tau\wedge t}||_{\ell_1^N(\mathbb{H})} \geq 1-\varepsilon\}.$$

But $\{||\tilde{Y}_{\tau\wedge s}||_{\ell_1^N(\mathbb{H})} \geq 1-\varepsilon\} \subset \{||\tilde{Y}_{\tau\wedge t}||_{\ell_1^N(\mathbb{H})} \geq 1-\varepsilon\}$ when $s \leq t$, so $\mathbb{P}(\tilde{Y}^* \geq 1) \leq \lim_{t\to\infty} \mathbb{P}(||\tilde{Y}_{\tau\wedge t}||_{\ell_1^N(\mathbb{H})} \geq 1-\varepsilon)$. Since $(\tilde{Y}_{\tau\wedge t})_{t\geq 0}$ takes values in the unit ball of $\ell_1^N(\mathbb{H})$ and is differentially subordinate to $(\tilde{X}_t)_{t\geq 0}$ (it is easy that this property is inherited from the pair (X, Y)), the previous lemma gives

$$\mathbb{P}(\tilde{Y}^* \geq 1) \leq \frac{4(\ln N+1)}{1-\varepsilon}||\tilde{X}||_{L_1(\ell_1^N(\mathbb{H}))},$$

and letting $\varepsilon \to 0$ yields $\mathbb{P}(\tilde{Y}^* \geq 1) \leq 4(\ln N+1)||\tilde{X}||_{L_1(\ell_1^N(\mathbb{H}))}$. Coming back to the processes X and Y, it remains to note that

$$\begin{aligned}\mathbb{P}(Y^* \geq 1) &\leq \mathbb{P}(|Y_0| \geq 1) + \mathbb{P}(\tilde{Y}^* \geq 1)\\ &\leq \mathbb{E}||X_0||_{\ell_1^N(\mathbb{H})} 1_{\{||Y_0||_{\ell_1^N(\mathbb{H})} \geq 1\}} + 4(\ln N+1)\sup_t \mathbb{E}||X_t||_{\ell_1^N(\mathbb{H})} 1_{\{||Y_0||_{\ell_1^N(\mathbb{H})} < 1\}}\\ &\leq 4(\ln N+1)\sup_t \mathbb{E}||X_t||_{\ell_1^N(\mathbb{H})}.\end{aligned}$$

□

Now we establish tight L_p estimates for differentially subordinate martingales.

Theorem 2.5. *Suppose that X, Y are martingales with values in $\ell_1^N(\mathbb{H})$, such that for each $j = 1, 2, \ldots, N$, the martingale Y^j is differentially subordinate to X^j. Then for any $1 < p < \infty$ we have*

$$||Y||_{L_p(\ell_1^N(\mathbb{H}))} \leq \frac{288p^2}{p-1}(\ln N + 1)||X||_{L_p(\ell_1^N(\mathbb{H}))}. \tag{2.5}$$

Proof. Fix numbers $\beta > 1$, $\lambda > 0$, $\delta > 0$ and consider the stopping times

$$\mu = \inf\{t \geq 0 : ||Y_t||_{\ell_1^N(\mathbb{H})} \geq \lambda\}, \quad \nu = \inf\{t \geq 0 : ||Y_t||_{\ell_1^N(\mathbb{H})} \geq \beta\lambda\}, \quad \sigma = \inf\{t \geq 0 : ||X_t||_{\ell_1^N(\mathbb{H})} > \delta\lambda\}.$$

Furthermore, introduce the processes

$$F_t = X_{\nu\wedge\sigma\wedge t} - X_{\mu\wedge\sigma\wedge t}, \qquad G_t = Y_{\nu\wedge\sigma\wedge t} - Y_{\mu\wedge\sigma\wedge t}.$$

Clearly, for any $j = 1, 2, \ldots, N$, the martingale G^j is differentially subordinate to F^j: if $s \in (\mu \wedge \sigma \wedge t, \nu \wedge \sigma \wedge t]$, then $d[G^j, G^j]_s = d[Y^j, Y^j]_s \leq d[X^j, X^j]_s = d[F^j, F^j]_s$; for remaining s, we have $d[G^j, G^j] = d[F^j, F^j] = 0$. We may write

$$\begin{aligned}
\mathbb{P}(Y^* > \beta\lambda,\, X^* \leq \delta\lambda) &\leq \mathbb{P}(\mu \leq \nu < \infty, \sigma = \infty) \\
&\leq \mathbb{P}(G^* > (\beta-1)\lambda) \leq \frac{4(\ln N + 1)}{(\beta-1)\lambda}||F||_{L_1(\ell_1^N(\mathbb{H}))} \leq \frac{8\delta(\ln N+1)}{\beta - 1}\mathbb{P}(Y^* \geq \lambda).
\end{aligned}$$

Here in the last line we have used the fact that $F = 0$ on $\{\mu = \infty\}$ and $|F_t| \leq 2\delta\lambda$ on $\{\mu < \infty\} \subseteq \{Y^* \geq \lambda\}$. Consequently, we see that

$$\mathbb{P}(Y^* \geq \beta\lambda) \leq \mathbb{P}(X^* \geq \delta\lambda) + \frac{8\delta(\ln N + 1)}{\beta - 1}\mathbb{P}(Y^* \geq \lambda).$$

Now, multiply both sides by $p\lambda^{p-1}$ and integrate over $(0, \infty)$ with respect to λ. We obtain

$$\beta^{-p}||Y^*||_{L_p}^p \leq \delta^{-p}||X^*||_{L_p}^p + \frac{8\delta(\ln N+1)}{\beta-1}||Y^*||_{L_p}^p,$$

which is equivalent to

$$||Y^*||_{L_p} \leq \delta^{-1}\left(\beta^{-p} - \frac{8\delta(\ln N + 1)}{\beta - 1}\right)^{-1/p}||X^*||_{L_p},$$

provided $\beta^{-p} > 8\delta(\ln N + 1)/(\beta - 1)$. Now we choose $\beta = 1+p^{-1}$ and $\delta = (32p(\ln N+1))^{-1}$, obtaining

$$||Y^*||_{L_p} \leq 32p(\ln N + 1)\left[\left(1 + \frac{1}{p}\right)^{-p} - \frac{1}{4}\right]^{-1/p}||X^*||_{L_p} \leq 288p(\ln N+1)||X^*||_{L_p},$$

since $\left(\left(1 + \frac{1}{p}\right)^{-p} - \frac{1}{4}\right)^{-1/p} \leq \left(\frac{1}{e} - \frac{1}{4}\right)^{-1} < 9$. To get the assertion, it remains to apply Doob's maximal inequality to the nonnegative submartingale $(||X_t||_{\ell_1^N(\mathbb{H})})_{t\geq 0}$. ☐

The above estimates for differentially subordinate martingales lead to inequalities for periodic Hilbert transform. We have the following.

Theorem 2.6. *For any $f \in L_1(\mathbb{T}; \ell_1^N(\mathbb{H}))$ we have the weak-type estimate*

$$|\{s \in \mathbb{T} : ||\mathcal{H}f(s)||_{\ell_1^N(\mathbb{H})} \geq 1\}| \leq 4(\ln N + 1)||f||_{L_1(\ell_1^N(\mathbb{H}))}.$$

Furthermore, for any $1 < p < \infty$ and any $f \in L_p(\mathbb{T}; \ell_1^N(\mathbb{H}))$ we have the strong-type bound

$$||\mathcal{H}f||_{L_p(\ell_1^N(\mathbb{H}))} \leq \frac{288p^2}{p-1}(\ln N + 1)||f||_{L_p(\ell_1^N(\mathbb{H}))}.$$

Proof. Take any function $f \in L_1(\mathbb{T};\ell_1^N(\mathbb{H}))$ and let u^j, v^j be the harmonic extensions of the j-th coordinate f^j of f and the Hilbert transform $\mathcal{H}f^j$, respectively. Consider the planar Brownian motion, starting at 0 and stopped at time $\tau = \inf\{t \geq 0 : |B_t| = 1\}$. Then the processes $X = (u(B_{\tau\wedge t}))_{t\geq 0}$, $Y = (v(B_{\tau\wedge t}))_{t\geq 0}$ are $\ell_1^N(\mathbb{H})$-valued martingales and for each j, Y^j is differentially subordinate to X^j: this follows at once from the identities

$$[X^j]_t = |f^j(0)|^2 + \int_{0+}^{\tau\wedge t} |\nabla u^j(B_s)|^2 \mathrm{d}s, \quad [Y^j]_t = |\mathcal{H}f^j(0)|^2 + \int_{0+}^{\tau\wedge t} |\nabla v^j(B_s)|^2 \mathrm{d}s = \int_{0+}^{t} |\nabla v^j(B_s)|^2 \mathrm{d}s$$

and the fact that u^j, v^j satisfy Cauchy-Riemann equations. Since the random variable B_τ is distributed uniformly on the unit circle, we have $||X||_{L_1(\ell_1^N(\mathbb{H}))} = ||f||_{L_1(\ell_1^N(\mathbb{H}))}$ and the inequality (2.4) implies

$$|\{\theta \in \mathbb{T} : ||\mathcal{H}f(e^{i\theta})||_{\ell_1^N(\mathbb{H})} \geq 1\}| \leq \mathbb{P}(Y^* \geq 1) \leq 4(\ln N + 1)||X||_{L_1(\ell_1^N(\mathbb{H}))} = 4(\ln N + 1)||f||_{L_1(\ell_1^N(\mathbb{H}))},$$

and similarly, by (2.5),

$$||\mathcal{H}f||_{L_p(\ell_1^N(\mathbb{H}))} = ||Y_\tau||_{L_p(\ell_1^N(\mathbb{H}))} \leq \frac{288p^2}{p-1}(\ln N + 1)||X||_{L_p(\ell_1^N(\mathbb{H}))} = \frac{288p^2}{p-1}(\ln N + 1)||f||_{L_p(\ell_1^N(\mathbb{H}))}. \square$$

3. An lower bound in (1.3) and (1.4)

In this section we will show that the growth of UMD constants for the Hilbert transform is logarithmic, even if $\mathbb{H} = \mathbb{R}$. It is convenient to split the material into a few parts.

3.1. **A bound for Hilbert transform implies bounds for analytic martingales.** Our starting point is the following fact taken from Hollenbeck, Kalton and Verbitsky [8] (see Theorem 2.3 there).

Theorem 3.1. *Let $F : \mathbb{C}^N \to \mathbb{R}$ be an upper semicontinuous function and let E be a nonempty subset of $\mathbb{C}^N$. In order that for every N-tuple $(P_1, P_2, \ldots, P_N)$ of polynomials with $(P_1(0), P_2(0), \ldots, P_N(0)) \in E$ we have*

$$\int_{-\pi}^{\pi} F(P_1(e^{i\theta}), P_2(e^{i\theta}), \ldots, P_N(e^{i\theta}))\frac{d\theta}{2\pi} \geq 0, \tag{3.1}$$

it is necessary and sufficient that there is a plurisubharmonic function $G : \mathbb{C}^N \to \mathbb{R}$ with $G \leq F$ and $G(z_1, z_2, \ldots, z_N) \geq 0$ for $(z_1, z_2, \ldots, z_N) \in E$.

We are ready to show the following fact.

Theorem 3.2. *Let N be a fixed positive integer. Then for any analytic martingale $Z = X + iY$ with values in $\mathbb{C}^N$ we have*

$$||Y||_{L_p(\ell_1^N(\mathbb{R}))} \leq ||\mathcal{H}||_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}||X||_{L_p(\ell_1^N(\mathbb{R}))} \tag{3.2}$$

and

$$\mathbb{P}(Y^* \geq 1) \leq ||\mathcal{H}||_{L_1(\ell_1^N(\mathbb{R}))\to L_{1,\infty}(\ell_1^N(\mathbb{R}))}||X||_{L_1(\ell_1^N(\mathbb{R}))}. \tag{3.3}$$

Proof. We will focus on (3.2), the proof of the inequality (3.3) is similar. We may assume that $||X||_{L_p(\ell_1^N(\mathbb{R}))} < \infty$, since otherwise there is nothing to prove. For the sake of clarity, we have decided to split the reasoning into a few intermediate steps.

Step 1. Application of Theorem 3.1. Denote the norm $||\mathcal{H}||_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}$ by c. Consider the continuous function $F : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$F(z_1, z_2, \ldots, z_N) = c^p\left(\sum_{k=1}^N |\Re z_k|\right)^p - \left(\sum_{k=1}^N |\Im z_k|\right)^p.$$

Let P_1, P_2, $\ldots$, P_N be arbitrary polynomials on $\mathbb{C}$ such that $\Im P_1(0) = \Im P_2(0) = \ldots = \Im P_N(0) = 0$. These functions are analytic in the unit disc, so if we define $f^j : \mathbb{T} \to \mathbb{R}$ by $f_j(e^{i\theta}) = \Re P_j(e^{i\theta})$, then we have $\mathcal{H} f_j = \Im P_j|_{\mathbb{T}}$. Consequently, (3.1) is equivalent to saying that

$$c^p ||(f_1, f_2, \ldots, f_N)||^p_{L_p(\ell_1^N(\mathbb{R}))} - ||(\mathcal{H} f_1, \mathcal{H} f_2, \ldots, \mathcal{H} f_N)||^p_{L_p(\ell_1^N(\mathbb{R}))} \geq 0,$$

which is obvious, by the very definition of the norm $||\mathcal{H}||_{L_p(\ell_1^N(\mathbb{R})) \to L_p(\ell_1^N(\mathbb{R}))}$. Therefore, Theorem 3.1 yields the existence of an appropriate plurisubharmonic function G on $\mathbb{C}^N$.

Step 2. A mollifying argument. Let g be a radial, C^∞ nonnegative function on $\mathbb{C}$, supported on the ball of center 0 and radius 1, satisfying $\int_{\mathbb{C}} g(z)\mathrm{d}z = 1$. For a fixed $\delta > 0$, define $\hat{G} = \hat{G}^\delta : \mathbb{C}^N \to \mathbb{R}$ by

$$\hat{G}(z) = \int_{\mathbb{C}^N} G(z_1 - u_1\delta, z_2 - u_2\delta, \ldots, z_N - u_N\delta) g(u_1) g(u_2) \ldots g(u_N) \mathrm{d}u.$$

This new function inherits most of the properties of G. First, it is clear that $\hat{G}$ is plurisubharmonic. It is also evident that $\hat{G}$ is of class C^∞. Next, we have $\hat{G} \geq G$ on $\mathbb{C}^N$, which follows from mean-value property of subharmonic functions and the fact that g is radial. Let us be a little more precise. Fix $k \in \{1, 2, \ldots, N-1\}$. Then, by the very definition of plurisubharmonicity, the function

$$\mathbb{C} \ni w \mapsto G(z_1, z_2, \ldots, z_{k-1}, w, z_{k+1} - u_{k+1}\delta, \ldots, z_N - u_N\delta)$$

is subharmonic. But g is a radial, nonnegative function of integral 1, so

$$\begin{aligned} &\int_{\mathbb{C}} G(z_1, z_2, \ldots, z_{k-1}, z_k - u_k\delta, z_{k+1} - u_{k+1}\delta, \ldots, z_N - u_N\delta) g(u_k) \mathrm{d}u_k \\ &\geq G(z_1, z_2, \ldots, z_{k-1}, z_k, z_{k+1} - u_{k+1}\delta, \ldots, z_N - u_N\delta) \end{aligned}$$

and hence

$$\begin{aligned} &\int_{\mathbb{C}^{N-k+1}} G(z_1, z_2, \ldots, z_{k-1}, z_k - u_k\delta, z_{k+1} - u_{k+1}\delta, \ldots, z_N - u_N\delta) \times \\ &\qquad\qquad \times g(u_k) g(u_{k+1}) \ldots g(u_N) \mathrm{d}u_k \mathrm{d}u_{k+1} \ldots \mathrm{d}u_N \\ &\geq \int_{\mathbb{C}^{N-k}} G(z_1, z_2, \ldots, z_{k-1}, z_k, z_{k+1} - u_{k+1}\delta, \ldots, z_N - u_N\delta) g(u_{k+1}) \ldots g(u_N) \mathrm{d}u_{k+1} \ldots \mathrm{d}u_N. \end{aligned}$$

This shows the majorization $\hat{G} \geq G$; in particular, we see that if $\Im z_1 = \Im z_2 = \ldots = \Im z_N = 0$, then $\hat{G}(z_1, z_2, \ldots, z_N) \geq G(z_1, z_2, \ldots, z_N) \geq 0$. We turn our attention to the last property of $\hat{G}$. Note that the inequality $G \leq F$ implies

$$\begin{aligned} \hat{G}(z) &\leq \int_{\mathbb{C}^N} F(z_1 - u_1\delta, z_2 - u_2\delta, \ldots, z_N - u_N\delta) g(u_1) g(u_2) \ldots g(u_N) \mathrm{d}u \\ &\leq c^p \left(\sum_{k=1}^N |\Re z_k| + N\delta \right)^p - \left(\sum_{k=1}^N \big| |\Im z_k| - \delta \big| \right)^p. \end{aligned} \tag{3.4}$$

Step 3. Application of Itô's formula. Now, fix an analytic martingale $Z = (Z_t)_{t \geq 0}$ with values in $\mathbb{C}^N$, starting from the set $E = \{z \in \mathbb{C}^N : \Im z_1 = \Im z_2 = \ldots = \Im z_N = 0\}$. For a given positive number M, consider the stopping time $\tau_M = \inf\{t \geq 0 : |Z_t| \geq M\}$. Since $\hat{G}$ is of class C^∞, we have

$$\hat{G}(Z_{\tau_M \wedge t}) = I_0 + I_1 + I_2/2, \tag{3.5}$$

where

$$I_0 = \hat{G}(Z_0), \quad I_1 = \int_{0+}^{\tau_M \wedge t} \hat{G}_z(Z_s) \mathrm{d}Z_s + \int_{0+}^{\tau_M \wedge t} \hat{G}_{\overline{z}}(Z_s) \mathrm{d}\overline{Z}_s, \quad I_2 = \sum_{j,k=1}^N \int_{0+}^{\tau_M \wedge t} \hat{G}_{z_j \overline{z_k}}(Z_s) \mathrm{d}[Z^j, \overline{Z^k}]_s.$$

Note that $\mathbb{E}I_1 = 0$ by the properties of stochastic integrals: indeed, the processes $(\hat{G}_z(Z_s))_{s\leq\tau_M}$, $(\hat{G}_{\overline{z}}(Z_s))_{s\leq\tau_M}$ are bounded on the set $\{\tau_M > 0\}$, so the integrals in I_1 define L^2-bounded martingales. To deal with the term I_2, observe that for any $h \in \mathbb{C}^N$ we have, by plurisubharmonicity of $\hat{G}$,

$$\sum_{j,k=1}^N \hat{G}_{z_j\overline{z_k}}(z)h_j\overline{h_k} \geq 0. \tag{3.6}$$

Fix $s < s_1 \leq t$ and for each n, let $(T_r^n)_{1\leq r\leq r_n}$ be a nondecreasing sequence of finite stopping times with $T_1^n = s$ and $T_{r_n}^n = s_1$, satisfying $\lim_{n\to\infty}\max_{1\leq r\leq r_n}|T_{r+1}^n - T_r^n| = 0$. Apply (3.6) to $z = Z_{\tau_M\wedge s}$ and $h = Z_{\tau_M\wedge T_{r+1}^n} - Z_{\tau_M\wedge T_r^n}$ for each $r = 1, 2, \ldots, r_n$. Summing over r and letting $n\to\infty$ gives

$$\sum_{j,k=1}^N \hat{G}_{z_j\overline{z_k}}(Z_{\tau_M\wedge s})[Z^j, \overline{Z^k}]_{\tau_M\wedge s}^{\tau_M\wedge s_1} \geq 0.$$

This yields $I_2 \geq 0$: simply approximate the integrals by discrete sums. Thus, combining the above facts with (3.5) gives $\mathbb{E}\hat{G}(Z_{\tau_M\wedge t}) \geq \mathbb{E}\hat{G}(Z_0)$. However, the right-hand side is nonnegative: this follows from the assumption $\Im Z_0^1 = \Im Z_0^2 = \ldots = \Im Z_0^N = 0$ and the properties of $\hat{G}$, see Step 2 above.

Step 4. Limiting arguments. We have shown that $\mathbb{E}\hat{G}(Z_{\tau_M\wedge t}) \geq 0$. By (3.4), we get

$$\mathbb{E}\left(\sum_{k=1}^N \big||Y_{\tau_M\wedge t}| - \delta\big|\right)^p \leq c^p\mathbb{E}\left(\sum_{k=1}^N |X_{\tau_M\wedge t}^k| + N\delta\right)^p,$$

which implies, after taking p-th root of both sides,

$$\left[\mathbb{E}\left(\sum_{k=1}^N \big||Y_{\tau_M\wedge t}| - \delta\big|\right)^p\right]^{1/p} \leq c\big(||X||_{L_p(\ell_1^N)} + N\delta\big).$$

It suffices to let $\delta\to 0$ and then $M\to\infty$ to get the claim, by virtue of Fatou's lemma. □

The next step is to show that the validity of (3.2) implies an appropriate weak-type (1,1) version.

Theorem 3.3. *Let N be a fixed positive integer. Then for any analytic martingale $Z = X + iY$ with values in $\mathbb{C}^N$ we have*

$$\mathbb{P}\left(Y^* \geq 1\right) \leq 2||\mathcal{H}||_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}||X||_{L_1(\ell_1^N(\mathbb{R}))}. \tag{3.7}$$

Proof. The assertion will follow from (3.2) via an appropriate stopping-time argument. Fix a small $\varepsilon > 0$ and consider the stopping times

$$\tau = \inf\{t\geq 0 : ||Y_t||_{\ell_1^N(\mathbb{R})} \geq 1-\varepsilon\}, \qquad \sigma = \inf\left\{t\geq 0 : ||X_t||_{\ell_1^N(\mathbb{R})} \geq ||\mathcal{H}||^{-1}_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}\right\}.$$

We have

$$\mathbb{P}\left(Y^*\geq 1\right) \leq \mathbb{P}\left(X^* > ||\mathcal{H}||^{-1}_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}\right) + \mathbb{P}\left(Y^*\geq 1,\, X^* \leq ||\mathcal{H}||^{-1}_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}\right). \tag{3.8}$$

By Doob's weak-type inequality for submartingales, we have

$$\mathbb{P}\left(X^* > ||\mathcal{H}||^{-1}_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}\right) \leq ||\mathcal{H}||_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}||X||_{L_1(\ell_1^N(\mathbb{R}))}.$$

To deal with the second probability, introduce the martingales $\tilde{X}_t = X_{\tau\wedge\sigma\wedge t}1_{\{\tau>0\}}$, $\tilde{Y}_t = Y_{\tau\wedge\sigma\wedge t}1_{\{\tau>0\}}$, for $t\geq 0$. Clearly, the martingale $\tilde{Z} = \tilde{X} + i\tilde{Y}$ is analytic. Furthermore, we have

$$\mathbb{P}\left(Y^*\geq 1,\, X^*\leq ||\mathcal{H}||^{-1}_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}\right) \leq \mathbb{P}(\tilde{Y}_t \geq 1-\varepsilon \text{ for some } t).$$

But $\{\tilde{Y}_t \geq 1-\varepsilon\} \subseteq \{\tilde{Y}_s \geq 1-\varepsilon\}$ when $t \leq s$, so $\mathbb{P}(\tilde{Y}_t \geq 1-\varepsilon \text{ for some } t) = \lim_{t\to\infty} \mathbb{P}(\tilde{Y}_t \geq 1-\varepsilon)$. Now, Chebyshev's inequality and (3.2) imply

$$\mathbb{P}(\tilde{Y}_t \geq 1-\varepsilon) \leq (1-\varepsilon)^{-p}||\tilde{Y}_t||^p_{L_p(\ell_1^N(\mathbb{R}))} \leq (1-\varepsilon)^{-p}||\mathcal{H}||^p_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}||\tilde{X}||^p_{L_p(\ell_1^N(\mathbb{R}))}.$$

But by the definition of the stopping time σ, the $\ell_1^N(\mathbb{R})$-norm of the martingale $\tilde{X}$ is bounded by $||\mathcal{H}||^{-1}_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}$, so we have the estimate $||\tilde{X}||^p_{L_p(\ell_1^N(\mathbb{R}))} \leq ||\mathcal{H}||^{1-p}_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}||\tilde{X}||_{L_1(\ell_1^N(\mathbb{R}))}$. Combining all the above facts, we obtain

$$\mathbb{P}\left(Y^* \geq 1,\ X^* \leq ||\mathcal{H}||^{-1}_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}\right) \leq (1-\varepsilon)^{-p}||\mathcal{H}||_{L_p(\ell_1^N(\mathbb{R}))\to L_p(\ell_1^N(\mathbb{R}))}||\tilde{X}||_{L_1(\ell_1^N(\mathbb{R}))}.$$

Plugging all the above observations into (3.8) and letting $\varepsilon \to 0$, we obtain the desired estimate. $\square$

In the light of (3.3) and (3.7), we see that the desired lower bounds in (1.3) and (1.4) are immediate consequences of the following statement.

Theorem 3.4. *Let N be a positive integer. Then there is an analytic martingale $Z = X + iY$ with values in $\mathbb{C}^N$ such that $\mathbb{P}(Y^* \geq 1) = 1$ and $||X||_{L_1(\ell_1^N(\mathbb{R}))} \leq \pi/(\ln N + 1)$.*

This theorem will be handled in two subsections below.

3.2. **A planar example.** We first provide the proper construction for $N = 1$. Let $Z = X + iY$ be the planar Brownian motion starting from $(0,0)$ and stopped upon hitting the set $\{0\}\times\big((-\infty,-1]\cup[1,\infty)\big)$. We will provide two facts concerning the random variables X_∞ and Y_∞.

Lemma 3.5. *For any $\lambda \geq 1$ we have* $\mathbb{P}(|Y_\infty| \geq \lambda) = \dfrac{4}{\pi}\arctan\big(\lambda - \sqrt{\lambda^2-1}\big)$.

Proof. Consider the conformal map $f(z) = i(z+z^{-1})/2$, which sends the upper halfplane $\mathbb{R}\times[0,\infty)$ onto the set $(\mathbb{C}\setminus i\mathbb{R})\cup i(-1,1)$. By Lévy's theorem, the composition $W = (f^{-1}(Z_t))_{t\geq 0}$ is an analytic martingale; this new process starts from i and terminates upon hitting the real axis. It is well-known that the law of W_∞ is the Cauchy distribution; consequently,

$$\begin{aligned}\mathbb{P}(|Y_\infty| \geq \lambda) = \mathbb{P}(|Z_\infty| \geq \lambda) &= \mathbb{P}(|W_\infty| \geq \lambda + \sqrt{\lambda^2-1}) + \mathbb{P}(|W_\infty| < \lambda - \sqrt{\lambda^2-1})\\ &= 1 - \frac{2}{\pi}\arctan(\lambda+\sqrt{\lambda^2-1}) + \frac{2}{\pi}\arctan(\lambda-\sqrt{\lambda^2-1})\\ &= \frac{4}{\pi}\arctan\big(\lambda-\sqrt{\lambda^2-1}\big).\end{aligned}$$

$\square$

Lemma 3.6. *We have $||X||_{L^1} = 1$.*

Proof. Let f, W be the conformal map and the analytic martingale studied in the previous lemma. Consider the stopping time $\tau_R = \inf\{t : |W_t| \in \{R^{-1}, R\}\}$, where $R > 1$ is a fixed parameter (with the standard convention $\inf\emptyset = +\infty$). Let $\eta(R)$ be an "error term" given by

$$\mathbb{E}|X_{\tau_R}| = \mathbb{E}\left|\Im\frac{W_{\tau_R}+W_{\tau_R}^{-1}}{2}\right| = \mathbb{E}\left|\Im\frac{W_{\tau_R}}{2}\right| + \mathbb{E}\left|\Im\frac{W_{\tau_R}^{-1}}{2}\right| + \eta(R).$$

Clearly, $\eta(R) \to 0$ as $R \to \infty$. Indeed, if $\tau_R = \infty$, then W_{τ_R} lies on the real line, so the imaginary parts of $(W_{\tau_R}+W_{\tau_R}^{-1})/2$, W_{τ_R}, $W_{\tau_R}^{-1}$ vanish, making no contribution to $\eta(R)$. If $|W_{\tau_R}| = R$, then

$$\left|\left|\Im\frac{W_{\tau_R}+W_{\tau_R}^{-1}}{2}\right| - \left|\Im\frac{W_{\tau_R}}{2}\right| - \left|\Im\frac{W_{\tau_R}^{-1}}{2}\right|\right| \leq |W_{\tau_R}|^{-1} = R^{-1}$$

and similarly, if $|W_{\tau_R}|=R^{-1}$, then $\left|\left|\Im\frac{W_{\tau_R}+W_{\tau_R}^{-1}}{2}\right|-\left|\Im\frac{W_{\tau_R}}{2}\right|-\left|\Im\frac{W_{\tau_R}^{-1}}{2}\right|\right|\leq|W_{\tau_R}|=R^{-1}$. Furthermore, the martingales $(W_{\tau_R\wedge t})_{t\geq 0}$, $(W^{-1}_{\tau_R\wedge t})_{t\geq 0}$ are bounded, so

$$\mathbb{E}\left|\Im\frac{W_{\tau_R}}{2}\right|=\mathbb{E}\Im\frac{W_{\tau_R}}{2}=\frac{1}{2},\qquad \mathbb{E}\left|\Im\frac{W_{\tau_R}^{-1}}{2}\right|=-\mathbb{E}\Im\frac{W_{\tau_R}^{-1}}{2}=\frac{1}{2}.$$

This proves that $\mathbb{E}|X_{\tau_R}|=1+\eta(R)$. It remains to perform appropriate limiting arguments. First, note that by Fatou's lemma we have, for any $R>1$,

$$||X||_{L^1}=\lim_{t\to\infty}||X_t||_1\geq\liminf_{t\to\infty}||X_{\tau_R\wedge t}||_{L^1}=||X_{\tau_R}||_{L^1}=1+\eta(R),$$

and hence $||X||_{L^1}\geq 1$. In addition, again by Fatou's lemma, $||X_t||_{L^1}\leq\liminf_{R\to\infty}||X_{\tau_R\wedge t}||_{L^1}\leq\liminf_{R\to\infty}||X_{\tau_R}||_{L^1}=1$ for any $t\geq 0$. This proves the claim. □

3.3. **An analytic martingale with values in $\mathbb{C}^N$.** The previous example, combined with an appropriate inductive argument, leads to the efficient $\ell_1^N(\mathbb{C})$-valued analytic martingale.

Theorem 3.7. *Let $\zeta=(\zeta_n)_{n\geq 1}$ be the sequence defined by $\zeta_1=1$ and the recursion*

$$\zeta_n=\frac{1}{n}+\zeta_{n-1}\left(1-\frac{2}{\pi n}\arctan\sqrt{n^2-1}-\frac{2\operatorname{arccosh}n}{\pi n}\right).$$

Then for any $N\geq 1$ there is an analytic martingale $W=U+iV$ with values in $\ell_1^N(\mathbb{C})$ satisfying $\mathbb{P}(V^*\geq 1)=1$ and $||U||_{L_1(\ell_1^N(\mathbb{C}))}=\zeta_N$.

Proof. If $N=1$, we set W to be the example from the previous section; the required properties of this process follow from Lemmas 3.5 and 3.6. Suppose that $N\geq 2$ and fix $\eta>0$. Let $\tilde{W}$ be the process corresponding to $N-1$ and the number $\eta/2$, guaranteed by the inductive assumption. Consider the analytic martingale $Z=X+iY$ of §3.2, independent of $\tilde{W}$ and let σ be the lifetime of Z, i.e., $\sigma=\inf\{t: Z_t\in\{0\}\times((-\infty,-1]\cup[1,\infty))\}$. We define W by the following formula: if $t\leq\sigma$, set $W_t=(Z_t/N,\underbrace{0,0,\ldots,0}_{N-1\text{ times}})$. On the other hand, if $t>\sigma$, set $W_t=(Z_\sigma/N,(1-|Y_\sigma/N|)_+\tilde{W}_{t-\sigma})$.
This process is analytic, since Z and $\tilde{W}$ were independent. Note that $V^*\geq 1$ almost surely. Indeed, if $|Y_\sigma|\geq N$, then the first coordinate of V already guarantees this estimate; on the other hand, if $|Y_\sigma|<N$, then, by the inductive assumption, $V^*\geq|Y_\sigma|/N+(1-|Y_\sigma/N|)\tilde{V}^*\geq 1$ with probability 1. To compute the first norm of U, observe that Lemma 3.5 implies that the density of the distribution of $|Y_\sigma/N|$ is equal to $g(\lambda)=\frac{2}{\pi\lambda\sqrt{\lambda^2N^2-1}}1_{[N^{-1},\infty)}(\lambda)$. Consequently,

$$\begin{aligned}||U||_{L_1(\ell_1^N(\mathbb{R}))}&=||U^1||_1+||(1-|Y_\sigma/N|)_+\tilde{U}||_{L_1(\ell_1^{N-1}(\mathbb{R}))}\\&=\frac{1}{N}+\int_{1/N}^1\frac{2(1-\lambda)}{\pi\lambda\sqrt{\lambda^2N^2-1}}\cdot\zeta_{N-1}\mathrm{d}\lambda\\&=\frac{1}{N}+\zeta_{N-1}\left(1-\frac{4}{\pi}\arctan(N-\sqrt{N^2-1})-\frac{2\operatorname{arccosh}N}{\pi N}\right)=\zeta_N.\end{aligned}$$ □

Observe that the statement above will imply the desired Theorem 3.4 once we show the following bound for the constant ζ_N.

Lemma 3.8. *For any $N\geq 1$ we have $\zeta_N\leq\pi/(\ln N+1)$.*

Proof. We will use induction. We have $\zeta_1 = 1 \leq \pi$ and

$$\zeta_2 = \frac{2}{\pi}\arctan\sqrt{3} - \frac{\operatorname{arccosh} 2}{\pi} + \frac{1}{2} \leq \frac{2}{\pi}\arctan\sqrt{3} + \frac{1}{2} = \frac{7}{6} \leq \frac{\pi}{\ln 2 + 1},$$

so the claim holds true for $N = 1$ and $N = 2$. Fix $N \geq 3$ and assume that it holds for $N - 1$. We have

$$\zeta_N \leq \frac{\pi}{\ln(N-1)+1}\left(\frac{2}{\pi}\arctan\sqrt{N^2-1} - \frac{2\ln N}{\pi N}\right) + \frac{1}{N},$$

since $\operatorname{arccosh} N \geq \ln N$. However, because $\dfrac{-2\ln N}{N(\ln(N-1)+1)} + \dfrac{1}{N} \leq \dfrac{-\ln(N-1)-1}{N(\ln(N-1)+1)} + \dfrac{1}{N} = 0$, it suffices to show that $\dfrac{2\arctan\sqrt{N^2-1}}{\ln(N-1)+1} \leq \dfrac{\pi}{\ln N + 1}$, or, equivalently,

$$(\ln N + 1)\cdot\frac{2}{\pi}\arctan\sqrt{N^2-1} \leq \ln(N-1) + 1. \tag{3.9}$$

Since $N \geq 3$, we have $\sqrt{N^2-1} \geq \sqrt{8}$ and hence $\arctan\sqrt{N^2-1} \leq \dfrac{\pi}{2} - \dfrac{8}{9\sqrt{N^2-1}}$ (indeed: an easy analysis of a derivative shows that the function $x \mapsto \arctan x - \frac{\pi}{2} + \frac{8}{9x}$ is increasing on $[\sqrt{8}, \infty)$ and vanishes at infinity). Plugging this into (3.9), we see that it is enough to prove that

$$\sqrt{N^2-1}\,\ln\frac{N}{N-1} \leq \frac{16}{9\pi}(\ln N + 1).$$

However, this inequality holds for $N = 3$ (the left-hand side is equal to $1.146829\ldots$, the right-hand side is equal to $1.187572\ldots$). Furthermore, the right-hand side is an increasing function of N, while the left-hand side decreases with N (a standard analysis shows that the left-hand side, considered as a function of $N \in [3, \infty)$, is convex and its derivative vanishes at infinity). This completes the proof. $\square$

Acknowledgment

The author would like to express his gratitude to anonymous referees for the careful reading of the manuscript and many helpful comments. The research was supported by Narodowe Centrum Nauki (Poland) grant DEC-2014/14/E/ST1/00532.

References

[1] R. Bañuelos and G. Wang, *Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms*, Duke Math. J. 80 (1995), 575–600.
[2] S. Bochner and A. E. Taylor, *Linear functionals on certain spaces of abstractly-valued functions*, Ann. of Math. (2) **39** (1938), no. 4, pp. 913–944.
[3] J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat. **21** (1983), 163–168.
[4] D. L. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, Conference on Harmonic Analysis in Honor of Antoni Zygmund, Chicago, 1981, Wadsworth, Belmont, CA (1983), 270–286.
[5] D. L. Burkholder, *Explorations in martingale theory and its applications*, École d'Ete de Probabilités de Saint-Flour XIX—1989, pp. 1–66, Lecture Notes in Math., 1464, Springer, Berlin, 1991.
[6] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. **124** (1970), 249–304.
[7] C. Dellacherie and P. A. Meyer, *Probabilities and potential B*, North-Holland, Amsterdam, 1982.
[8] B. Hollenbeck, N. J. Kalton and I. E. Verbitsky, *Best constants for some operators associated with the Fourier and Hilbert transforms*, Studia Math. **157** (2003), 237–278.

[9] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis, Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 63. Springer, Cham, 2016.

[10] G. Wang, *Differential subordination and strong differential subordination for continuous time martingales and related sharp inequalities*, Ann. Probab. 23 (1995), 522–551.

[11] A. Zygmund, *Trigonometric series* Vol. 2, Cambridge University Press, London, 1968.

Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

Email address: `ados@mimuw.edu.pl`