A sharp maximal inequality for one-dimensional Dunkl martingales

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Abstract

Let $X = (X_t)_{t\geq 0}$ be a one-dimensional Dunkl process of parameter $k \geq 0$, starting from 0. For any $p \geq 1$, we find the least constant $C_{p,k} \in (0, \infty]$ in the Doob-type inequality

$$\mathbb{E}\left(\sup_{0\leq t\leq \tau} X_\tau\right)^p \leq C_{p,k}\mathbb{E}|X_\tau|^p$$

where τ runs over all $p/2$-integrable stopping times of X. The proof exploits optimal stopping techniques.

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1. Introduction

The purpose of this paper is to study a natural question about one-dimensional Dunkl martingales, important processes which originate from problems of mathematical physics. Suppose that $k \geq 0$ is a fixed number. A real-valued Feller process $\{(X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}}\}$ is called *a Dunkl process of parameter* k, if its infinitesimal generator L_k equals

$$L_k f(x) = \frac{1}{2}f''(x) + k\left(\frac{f'(x)}{x} - \frac{f(x) - f(-x)}{2x^2}\right)$$

for $f \in C^2(\mathbb{R})$. These processes, introduced and studied by Rösler (1998) and Rösler and Voit (1998), originate from the analytic work by Dunkl (1992) concerning special functions associated with root systems. Dunkl processes may be regarded as a generalization of the classical Brownian motion (which

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corresponds to the choice $k=0$). To make this connection more clear, note that the operator $2L_k$ is the square of the first order differential-difference operator

$$T_k f(x) = f'(x) + k\frac{f(x) - f(-x)}{x}, \quad f \in C^1(\mathbb{R}),$$

which can be thought of as a perturbed derivation operator. We should mention here that the papers of Dunkl (1989, 1992) actually motivate the more general class of processes with values in $\mathbb{R}^d$ (which were studied in depth in Rösler (1998); Rösler and Voit (1998)), but the problem presented in this paper makes sense only in the one-dimensional case.

Dunkl processes enjoy many interesting properties, and we will briefly recall some of them. First, for any $x \in \mathbb{R}$, the process $(X_t)_{t\geq 0}$ is a martingale under $\mathbb{P}_x$. Actually, as shown by Gallardo and Yor (2006), the Dunkl process $(X_t)_{t\geq 0}$ of parameter k is the unique martingale whose absolute value is a Bessel process of dimension $1+2k$. In addition, we have the following Brownian scaling property:

$$\{(X_{ct})_{t\geq 0}, \mathbb{P}_x\} \overset{(d)}{=} \{(\sqrt{c}X_t)_{t\geq 0}, \mathbb{P}_{x/\sqrt{c}}\}, \quad c > 0. \tag{1}$$

We also mention here that the Dunkl processes are discontinuous when $k > 0$; they can be realized as the sum of a Brownian motion and a pure-jump martingale (cf. Gallardo and Yor (2006)).

We will be particularly interested in Doob-type maximal inequalities for Dunkl processes. If $B = (B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion starting from 0, then for any $1 < p < \infty$ and any stopping time $\tau \in L^{p/2}$, we have

$$\mathbb{E}\left(\sup_{0\leq t\leq \tau} |B_t|\right)^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|B_\tau|^p$$

and the constant $(p/(p-1))^p$ is the best possible (cf. Wang (1991)). As it turns out, the constant is already optimal in the weaker one-sided bound

$$\mathbb{E}\left(\sup_{0\leq t\leq \tau} B_t\right)^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|B_\tau|^p.$$

There is a natural question about sharp versions of the above estimates for Dunkl martingales of parameter $k > 0$, starting from 0. As we have mentioned above, if X is such a process, then $|X|$ is a Bessel process of dimension $1+2k$. Maximal bounds for Bessel processes were studied by Pedersen (2000): using the results of that paper, we get that for any $1 \leq p < \infty$ and any stopping time $\tau \in L^{p/2}$, we have the sharp bound

$$\mathbb{E}\left(\sup_{0\leq t\leq \tau} |X_t|\right)^p \leq \left(\frac{p}{p-1+2k}\right)^{p/(1-2k)} \mathbb{E}|X_\tau|^p.$$

It may be a little unexpected that for the one-sided version the constant changes. For convenience, set

$$r_\pm = \frac{1-2k \pm \sqrt{4k^2+1}}{2}. \tag{2}$$

Clearly, $r_- \leq 0 < r_+ \leq 1$. Here is our main result.

Theorem 1.1. *Suppose that $X = (X_t)_{t\geq 0}$ is a Dunkl process of parameter $k > 0$, starting from 0. Then for any $1 \leq p < \infty$ and any stopping time $\tau \in L^{p/2}$ of X, we have*

$$\mathbb{E}\left(\sup_{0\leq t\leq \tau} X_t\right)^p \leq C_{p,k}\mathbb{E}|X_\tau|^p, \tag{3}$$

where the best constant is given by

$$C_{p,k} = (p-1+2k)^{-1}\frac{(p-r_-)^{(p-r_-)/(r_+-r_-)}}{(p-r_+)^{(p-r_+)/(r_+-r_-)}}. \tag{4}$$

The proof of this fact will rest on the optimal stopping techniques. To the best of our knowledge, this is one of the very few cases in which the solution to an optimal stopping problem for continuous-time discontinuous-path processes can be expressed in terms of compact and elementary formulas. We will actually show several important extensions of (3), including those corresponding to different starting points of the process $(X_t, \sup_{0\leq s\leq t} X_s)$. It turns out that the almost-optimal stopping times will be of the form $\tau_m = \inf\{t \geq 0 : X_t \leq m\sup_{0\leq s\leq t} X_s\}$. As a by product, we will obtain certain sharp integrability properties of this family of stopping times, thus generalizing the corresponding results of Pedersen (2000) and Wang (1991).

2. An optimal stopping problem

Throughout this section, the parameters p and k are fixed. As we have already mentioned, our approach exploits optimal stopping techniques (a convenient reference is the book by Peskir and Shiryaev (2005)). For brevity, we will use the notation $Y_t = \max\left\{\sup_{0\leq s\leq t} X_s, 0\right\}$. Then the main inequality (3) can be rewritten in the form

$$\sup_{\tau\in L^{p/2}} \mathbb{E}_0\left\{Y_\tau^p - C_{p,k}|X_\tau|^p\right\} \leq 0. \tag{5}$$

The first step in the analysis of this estimate is to extend it so that the process (X, Y) can start at arbitrary points of the domain $\mathfrak{D} = \{(x, y) \in \mathbb{R}\times[0,\infty) : x \leq y\}$. This is straightforward: for $(x, y) \in \mathfrak{D}$, the couple $(X_t, Y_t)^{(x,y)} \stackrel{\text{def}}{=} (X_t, Y_t\vee y)$ starts from (x, y) and is Markov under $\mathbb{P}_x$ (here $a \vee b = \max\{a, b\}$). So, $\mathbb{P}_{x,y} = \text{Law}((X_t, Y_t)^{(x,y)}|\mathbb{P}_x)$, $(x, y) \in \mathfrak{D}$, forms a Markovian family of probability measures on the canonical space. Therefore, we can extend the problem (5) to

$$U(x, y) = \sup_\tau \mathbb{E}_{x,y}\left\{Y_\tau^p - C_{p,k}|X_\tau|^p\right\},$$

where the supremum is taken over all stopping times τ satisfying $\mathbb{E}\tau^{p/2} < \infty$. In fact, we will study the more general problem involving an arbitrary constant $c > 0$:

$$U_c(x, y) = \sup_\tau \mathbb{E}_{x,y}\left\{Y_\tau^p - c|X_\tau|^p\right\}. \tag{6}$$

The remaining analysis is split into eight separate parts. The first four of them are heuristic: the arguments presented there lead us to an explicit candidate for U_c; in the next four we show rigorously that the candidate is indeed the solution to (6).

1° In the first step we apply the Brownian scaling property (1) of the Dunkl martingale and the fact that $\tilde{\tau} = \tau/t$ is a stopping time for the Dunkl process $s \mapsto t^{-1/2} X_{ts}$. We have

$$\begin{aligned} t^{-p/2} & \sup_{\tau} \mathbb{E}_x \Big\{ Y_\tau^p \vee y^p - c|X_\tau|^p \Big\} \\ & = \sup_{\tau/t} \mathbb{E}_{\frac{x}{\sqrt{t}}} \left\{ \left(\frac{Y_{t(\tau/t)}}{\sqrt{t}} \right)^p \vee \left(\frac{y}{\sqrt{t}} \right)^p - c \left| \frac{X_{t(\tau/t)}}{\sqrt{t}} \right|^p \right\}, \end{aligned}$$

from which it follows that $U_c(x,y) = y^p U_c(x/y, 1)$ for all $(x,y) \in \mathfrak{D}$, $y > 0$. Fix c and set $\varphi(s) = U_c(s,1)$ for $s \in (-\infty, 1)$. Thus $U_c(x,y) = y^p \varphi(x/y)$(for $y > 0$) and the problem boils down to finding the explicit formula for φ.

2° From the general theory of optimal stopping, the domain $\mathfrak{D}$ splits into two sets, the continuation set C and the stopping region D, given by

$$C = \{(x,y) : U_c(x,y) > |y|^p - c|x|^p\}, \quad D = \{(x,y) : U_c(x,y) = |y|^p - c|x|^p\}.$$

By the homogeneity property of U_c proved in the previous step, these two regions satisfy the condition $(x,y) \in C$ iff $(\lambda x, \lambda y) \in C$ for each $\lambda > 0$, and similarly for D. Furthermore, the stopping time which gives equality in (6), should be defined by

$$\tau_D = \inf\{t : (X_t, Y_t) \in D\}.$$

This reduces the problem (6) to determining the stopping set D and the value function U_c outside D. The following heuristic argument can be helpful in the search of these objects. Namely, when computing the supremum in (6), we want to make $\mathbb{E}_x Y_\tau^p \vee y^p$ large and, at the same time, keep $\mathbb{E}_x |X_\tau|^p$ relatively small. However, the expression $\mathbb{E}_x |X_t|^p$ increases when t increases (the process $(|X_t|^p)_{t \geq 0}$ is a submartingale), so to compensate this loss we need to make sure that $\mathbb{E}_x Y_t^p \vee y^p$ also increases substantially. This will be the case if x is close to y; otherwise, it will take too much time for X to get into proximity of y (and increase Y). In other words, it is natural to *conjecture* that the stopping region is of the form $D = \{(x,y) : x \leq my\}$ for some parameter $m = m(c) \in (0,1)$. Equipped with this observation, we use standard Markovian arguments (see Peskir and Shiryaev (2005)) and formulate the system

$$L_k U_c(\cdot, y) = 0 \qquad \text{for } y \geq 0,\ x \in (my, y), \tag{7}$$

$$\partial_y U_c(y-, y) = 0 \qquad \text{for } y \geq 0, \tag{8}$$

$$U_c(x,y) = y^p - c|x|^p \qquad \text{for } s \leq m, \tag{9}$$

$$\partial_x U_c(my, y) = -pc(my)^{p-1} \qquad \text{for } y \geq 0. \tag{10}$$

The conditions (7) and (8) just describe the action of the generator of (X, Y) on the function U_c; the equality (9) corresponds to instantaneous stopping, while (10) imposes the smooth-fit assumption.

3° To solve the system, we start with (7). It can be rewritten as $L_k\varphi = 0$, or

$$\varphi''(s) + k\left(\frac{2}{s}\varphi'(s) - \frac{\varphi(s) - \varphi(-s)}{s^2}\right) = 0,$$

for $s \in (m, 1)$. For such s we have $-s < m$ and hence (9) implies $\varphi(-s) = 1 - cs^p$. Thus we get

$$\varphi''(s) + \frac{2k\varphi'(s)}{s} - \frac{k\varphi(s)}{s^2} = \frac{k(cs^p - 1)}{s^2},$$

for $s \in (m, 1)$. We easily find a general solution to this equation. Indeed, the function

$$\lambda(s) = \frac{kcs^p}{p(p-1) + 2kp - k} + 1$$

is a particular solution. Furthermore, recalling the parameters r_+, r_- given by (2), we easily check that the functions $\varphi_+(s) = s^{r_+}$, $\varphi_-(s) = s^{r_-}$ form a fundamental set of solutions to the equation $\varphi''(s) + \frac{2k\varphi'(s)}{s} - \frac{k\varphi(s)}{s^2} = 0$, $s \in (m, 1)$. Hence, the desired function φ restricted to $[m, 1]$ must be of the form

$$\varphi(s) = \frac{kcs^p}{p(p-1) + 2kp - k} + 1 + \alpha s^{r_+} + \beta s^{r_-},$$

for some α, β to be found. To derive them, we exploit the equations $\varphi(m) = 1 - cm^p$ and $\varphi'(m) = -cpm^{p-1}$ (which follow directly from (9) and (10)), obtaining the system

$$\begin{aligned}
\frac{c(p(p-1) + 2kp)}{p(p-1) + 2kp - k} + \alpha m^{r_+ - p} + \beta m^{r_- - p} &= 0,\\
\frac{c(p(p-1) + 2kp)}{p(p-1) + 2kp - k} + \frac{\alpha r_+}{p} m^{r_+ - p} + \frac{\beta r_-}{p} m^{r_- - p} &= 0.
\end{aligned}$$

This is easily solved in α, β:

$$\begin{aligned}
\alpha_{c,m} &= \frac{pc(p - 1 + 2k)(p - r_-)m^{p - r_+}}{(p(p-1) + 2kp - k)(r_- - r_+)},\\
\beta_{c,m} &= \frac{pc(p - 1 + 2k)(p - r_+)m^{p - r_-}}{(p(p-1) + 2kp - k)(r_+ - r_-)}.
\end{aligned}$$

Plugging these values into the definition of φ and exploiting (8), we obtain

$$\frac{(p - 1 + 2k)(m^{p - r_+} - m^{p - r_-})}{r_+ - r_-} = c^{-1} \tag{11}$$

(we have $(p - r_-)(p - r_+) = p(p-1) + 2kp - k$).

4° Denote the left-hand side of (11) by $F(m)$ and put

$$m_{\max} = \left(\frac{p-r_+}{p-r_-}\right)^{1/(r_+-r_-)}.$$

One easily verifies that F is increasing on $(0, m_{\max})$ and decreasing on $(m_{\max}, 1)$; furthermore, we have $F(0) = F(1) = 0$ and $F(m_{\max}) = C_{p,k}^{-1}$. For $c \geq C_{p,k}$, let $m(c) \in (0,1)$ denote the largest root of (11). Thus, we are led to the function $\varphi = \varphi_{c,m(c)}$ given by

$$\frac{kcs^p}{p(p-1)+2kp-k} + 1 + \alpha_{c,m(c)} s^{r_+} + \beta_{c,m(c)} s^{r_-}$$

when $s \in (m(c), 1]$, and

$$\varphi_{c,m(c)} = 1 - c|s|^p \qquad \text{if } s \leq m(c).$$

We would also like to mention here that the above formula also makes sense if $m(c)$ is replaced by an arbitrary number $m > m(c)$. For consistence, such a function will be denoted by $\varphi_{c,m}$ and will play an important role in our considerations below.

Having found φ, we obtain the following solution to the system (7)-(10):

$$W_c(x,y) = \begin{cases} y^p \varphi_{c,m(c)}(x/y) & \text{if } y > 0, \\ y^p - c|x|^p & \text{if } y = 0. \end{cases} \tag{12}$$

Before we proceed, let us prove three useful facts about W_c. First, for each $y \geq 0$ we have

$$L_k W_c(\cdot, y) \leq 0 \quad \text{ on } (-\infty, 0) \cup (0, my) \tag{13}$$

or, equivalently, $L_k\varphi(s) \leq 0$ on $(-\infty, 0)\cup(0, m)$. Indeed, for $|s| < m$ we compute that $2L_k\varphi(s) = -cp(p-1+2k)|s|^{p-2} < 0$, while for $s \leq -m$,

$$\begin{aligned} 2L_k\varphi(s) = & -\frac{cp(p-1)(p-1+2k)(p+2k)|s|^{p-2}}{p(p-1)+2kp-k} \\ & + k|s|^{r_--2}(\alpha_{c,m(c)}|s|^{r_+-r_-} + \beta). \end{aligned}$$

Now, the first summand on the right is negative; moreover, if we set $G(s) = \alpha_{c,m(c)}|s|^{r_+-r_-} + \beta$, we easily check that

$$G(-m(c)) = -\frac{m(c)^{p-r_-} cp(p-1+2k)}{p(p-1)+2kp-k} < 0,$$

and that for $s < -m$, the derivative

$$G'(s) = \frac{cp(p-1+2k)(p-r_-)}{(p(p-1)+2kp-k)m(c)^{r_+-p}} |s|^{r_+-r_--1}$$

is positive. Thus $L_k\varphi \le 0$ also in this case and (13) follows. The next property of W_c is that

$$W_c(x,y) \ge |y|^p - c|x|^p \qquad \text{for } (x,y) \in \mathfrak{D} \tag{14}$$

or, equivalently, $\varphi(s) \ge 1 - c|s|^p$ for $s \le 1$. Clearly, it suffices to show the latter bound only for $s \ge m$. Consider the function $H(s) = s^{-p}(\varphi(s) - 1 + cs^p)$. Then the desired majorization follows from the equality $H(m) = 0$ and the fact that

$$H'(s) = \frac{cp(p-1+2k)}{(r_+ - r_-)s}\left((s/m)^{r_+ - p} - (s/m)^{r_- - p}\right)$$

is nonnegative.

Finally, fix $c \ge C_{p,k}$ and $m > m(c)$. Let $\varphi = \varphi_{c,m}$ be the function defined just above (12) and let $W_{c,m}$ be given by (12). Then we easily see that $W_{c,m}$ satisfies (7), (9) and (10); on contrary, the condition (8) (which corresponded to the equality (11)) does not hold; but this time, we have $\partial_y W_{c,m}(y-, y) = y^{p-1}(p\varphi(1) - \varphi'(1)) \ge 0$ for $y \ge 0$. Indeed: if $m > m(c)$, then the left-hand side of (11) is smaller than the right-hand side.

5° We turn to the rigorous verification that the functions U_c and W_c, given by (6) and (12), coincide. In this step we will prove the bound $U_c \le W_c$. Fix $\tau \in L^{p/2}$. We extend φ to the whole $\mathbb{R}$ by

$$\varphi(s) = \frac{kcs^p}{p(p-1) + 2kp - k} + 1 + \alpha_{c,m(c)}s^{r_+} + \beta_{c,m(c)}s^{r_-}$$

when $s > 1$. Then φ is of class C^1. So, if we extend W_c to the whole $\mathbb{R} \times [0, \infty)$ with the use of the formula (12), we easily see that this extension is of class C^1 on $\mathbb{R} \times (0, \infty)$ and of class C^2 on $\{(x,y) : y > 0,\ x/y \notin \{0, m\}\}$. But $L_k W_c(\cdot, y) \le 0$ on $\mathfrak{D} \setminus \{(x,y) : x/y \in \{0, m(c)\}\}$, by (7) and (13); in addition, W_c satisfies (8). Consequently, using Ito's formula, we get that the process $(W_c(X_{\tau\wedge t}, Y_{\tau\wedge t}))_{t\ge 0}$ is a local supermartingale. Actually, we should stress here that we do not use the classical form of Itô's formula (which requires an appropriate regularity of W_c), but rather its extension due to Peskir (2005). It enables to handle the fact that the C^2-property of W_c fails on the half-lines $\{(x,y) : x = m(c) \cdot y,\ y \ge 0\}$ and $\{(x,y) : x = 0,\ y \ge 0\}$. Let $(\sigma_n)_{n\ge 0}$ be a localizing sequence of $(W_c(X_{\tau\wedge t}, Y_{\tau\wedge t}))_{t\ge 0}$. By (14), we get

$$\mathbb{E}_{x,y}\left[Y^p_{\tau\wedge\sigma_n\wedge t} - c|X_{\tau\wedge\sigma_n\wedge t}|^p\right] \le \mathbb{E}_{x,y} W_c(X_{\tau\wedge\sigma_n\wedge t}, Y_{\tau\wedge\sigma_n\wedge t}) \le W_c(x,y),$$

for $t \ge 0$. Now, if $\tau \in L^{p/2}$, then $Y_\tau \in L^p$: this follows from Burkholder-Davis-Gundy inequalities for Bessel processes (see DeBlassie (1987)). Hence, letting $t \to \infty$, $n \to \infty$ and using Lebesgue dominated convergence theorem, we get $\mathbb{E}_{x,y}\left[Y^p_\tau - c|X_\tau|^p\right] \le W_c(x,y)$. Hence, since τ was arbitrary, the inequality $U_c(x,y) \le W_c(x,y)$ is established.

6° Fix $m > m_{\max} = m(C_{p,k})$ and consider the stopping time $\tau = \tau_m = \inf\{t : X_t \le mY_t\}$. The purpose of this step is to show that $\mathbb{E}_{x,y}\tau^{p/2} < \infty$ for all (x,y). Clearly, we may assume that $x > my$, since otherwise $\tau \equiv 0$ and there is nothing to prove. Let $\varphi = \varphi_{C_{p,k},m(C_{p,k})}$ and $\tilde{\varphi} = \varphi_{C_{p,k},m}$; denote by α, β, $\tilde{\alpha}$, $\tilde{\beta}$ the corresponding parameters appearing in the definitions of these functions. First we show that $\varphi > \tilde{\varphi}$ on $[m,1]$. This is equivalent to saying that $F(s) = (\alpha - \tilde{\alpha})s^{r_+ - r_-} + \beta - \tilde{\beta} \geq 0$, and follows from the inequality $\varphi(m) \geq 1 - C_{p,k}m^p = \tilde{\varphi}(m)$ (so $F(m) \geq 0$) and $F' > 0$ on $(m,1)$. The latter bound is a direct consequence of the estimates $r_+ - r_- > 0$ and $\alpha - \tilde{\alpha} > 0$, the second being equivalent to $m_{\max}^{p-r_+} < m^{p-r_+}$. Therefore, the functions $W(x,y) = y^p\varphi(x/y)$ and $\tilde{W}(x,y) = y^p\tilde{\varphi}(x/y)$ satisfy $W(x,y) \geq \tilde{W}(x,y)$ provided $x \geq my$; note that we also have this bound (actually, an equality) when $x \leq 0$. Moreover, as we have observed at the end of 4°, the function $\tilde{W}$ satisfies (7), (9), (10) and $\partial_y \tilde{W}(y,y) = y^{p-1}(p\tilde{\varphi}(1) - \tilde{\varphi}'(1)) \geq 0$. Thus, by Itô's formula,

$$\mathbb{E}_{x,y}\tilde{W}(X_{\tau\wedge\sigma_n\wedge t}, Y_{\tau\wedge\sigma_n\wedge t}) = \tilde{W}(x,y) + (\tilde{\varphi}(1) - p^{-1}\tilde{\varphi}'(1))(\mathbb{E}_{x,y}Y^p_{\tau\wedge\sigma_n\wedge t} - y^p),$$

for some localizing sequence $(\sigma_n)_{n\geq 0}$. Therefore, using the bound $W \geq \tilde{W}$ and Itô's formula again,

$$\begin{aligned}\tilde{W}(x,y) + (\tilde{\varphi}(1) - p^{-1}\tilde{\varphi}'(1))(\mathbb{E}_{x,y}Y^p_{\tau\wedge\sigma_n\wedge t} - y^p) &\leq \mathbb{E}_{x,y}W(X_{\tau\wedge\sigma_n\wedge t}, Y_{\tau\wedge\sigma_n\wedge t}) \\ &= W(x,y),\end{aligned}$$

which yields

$$\mathbb{E}_{x,y}Y^p_{\tau\wedge\sigma_n\wedge t} \leq y^p + \frac{W(x,y) - \tilde{W}(x,y)}{\tilde{\varphi}(1) - p^{-1}\tilde{\varphi}'(1)}.$$

This, by Lebesgue's monotone convergence theorem, implies $\mathbb{E}_{x,y}Y_\tau^p < \infty$ and hence, by Burkholder-Davis-Gundy inequality for Bessel processes, gives also that $\tau \in L^{p/2}$ for all (x,y).

7° We are ready to prove the bound $U_c(x,y) \geq W_c(x,y)$. This is trivial when $y = 0$ or $x/y \leq m$: simply take $\tau \equiv 0$ in the definition of U_c. Now, pick an arbitrary $c > C_{p,k}$; then $m = m(c) > m(C_{p,k})$ and the stopping time τ_m is $p/2$-integrable. Moreover, $\mathbb{E}Y^p_{\tau_m} < \infty$ and

$$U_c(x,y) \geq \mathbb{E}_{x,y}(Y^p_{\tau_m} - cX^p_{\tau_m}) = \mathbb{E}_{x,y}W_c(X_{\tau_m}, Y_{\tau_m}) = W_c(x,y),$$

where the latter equality follows from Itô's formula and the condition $\mathbb{E}Y^p_{\tau_m} < \infty$. Thus we are done provided $c > C_{p,k}$. To handle the case $c = C_{p,k}$, observe that $U_{C_{p,k}} \geq U_{c'}$ when $c' > C_{p,k}$ and $W_{c'} \to W_{C_{p,k}}$ as $c' \downarrow C_{p,k}$. This completes the proof of $U = W$.

8° Finally, we show two facts: the inequality (3) does not hold with any constant smaller than $C_{p,k}$ and the stopping time $\tau := \tau_{m_{\max}}$ does not belong to $L^{p/2}$ when $x > m_{\max}y$. We start with the second statement. Suppose that $\mathbb{E}_{x,y}\tau^{p/2} < \infty$. By Burkholder-Davis-Gundy inequality, we get $\mathbb{E}_{x,y}Y^p_\tau < \infty$ and hence, by Lebesgue's dominated convergence theorem,

$$U_c(x,y) = \mathbb{E}_{x,y}\big[Y^p_{\tau_{m(c)}} - c|X_{\tau_{m(c)}}|^p\big] \xrightarrow{c\to C_{p,k}} \mathbb{E}_{x,y}\big[Y^p_\tau - C_{p,k}|X_\tau|^p\big].$$

However, we have $U_c(x,y) \xrightarrow{c\to C_{p,k}} U_{C_{p,k}}(x,y)$. Thus, if $m > m(C_{p,k})$, then, using the notation of step 6°, we may apply Itô's formula to obtain

$$\begin{aligned} U_{C_{p,k}}(x,y) &= \mathbb{E}_{x,y}\left[Y_\tau^p - C_{p,k}|X_\tau|^p\right] \\ &\leq \tilde{U}(x,y) + (\tilde{\varphi}(1) - p^{-1}\tilde{\varphi}'(1))\left(\mathbb{E}_{x,y}Y_\tau^p - y^p\right). \end{aligned}$$

Consequently, we get

$$\mathbb{E}_{x,y}Y_\tau^p \geq \frac{U_{C_{p,k}}(x,y) - \tilde{U}(x,y)}{\tilde{\varphi}(1) - p^{-1}\tilde{\varphi}'(1)} + y^p.$$

It remains to note that if we let $m \downarrow m_{\max}$, then the numerator is of order $O(m - m_{\max})$, while the denominator is of order $O((m - m_{\max})^2)$; thus, the right-hand side converges to infinity. This contradicts the assumption $\mathbb{E}\tau_{m_{\max}}^{p/2} < \infty$ we imposed at the beginning.

Now, suppose that the inequality (3) holds with some constant $\tilde{c} < C_{p,k}$. Consider the stopping times $\tau = \inf\{t : |X_t| \geq 1\}$ and for a fixed $c > C_{p,k}$,

$$\sigma = \begin{cases} \inf\{t > \tau : X_t \leq m(c)Y_t\} & \text{if } X_\tau = 1, \\ \tau & \text{otherwise.} \end{cases}$$

By the above argumentation and Markov property, we get

$$\begin{aligned} 0 &\geq \mathbb{E}_{0,0}\left(Y_\sigma^p - \tilde{c}|X_\sigma|^p\right) \\ &= \mathbb{E}_{0,0}\left(Y_\sigma^p - \tilde{c}|X_\sigma|^p\right)1_{\{\sigma=\tau\}} + \mathbb{E}_{0,0}\left(Y_\sigma^p - \tilde{c}|X_\sigma|^p\right)1_{\{\sigma>\tau\}} \\ &\geq -\tilde{c}\mathbb{P}(\sigma = \tau) + \mathbb{E}_{1,1}(Y_{\tau_{m(c)}}^p - \tilde{c}|X_{\tau_{m(c)}}|^p) \cdot \mathbb{P}(\sigma > \tau) \\ &= -\tilde{c}\mathbb{P}(\sigma = \tau) + \mathbb{P}(\sigma > \tau)(U_c(1,1) + (c - \tilde{c})\mathbb{E}|X_{\tau_{m(c)}}|^p). \end{aligned}$$

Now, letting $c \downarrow C_{p,k}$, we see that $\mathbb{E}\tau_{m(c)}^{p/2} \to \infty$ and hence also $\mathbb{E}|X_{\tau_{m(c)}}|^p \to \infty$. Thus the right-hand side above explodes to ∞, and we get the contradiction. Hence (3) cannot hold with a constant smaller than $C_{p,k}$.

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