BURKHOLDER'S FUNCTION
AND A WEIGHTED L^2 BOUND FOR STOCHASTIC INTEGRALS

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ABSTRACT. Let X be a continuous-path martingale and let Y be a stochastic integral, with respect to X, of some predictable process with values in $[-1, 1]$. We provide an explicit formula for Burkholder's function associated with the weighted L^2 bound

$$\|Y\|_{L^2(W)} \lesssim [w]_{A_2} \|X\|_{L^2(W)}.$$

1. INTRODUCTION

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-algebras of $\mathcal{F}$. Throughout the paper, we will assume that all adapted martingales have continuous paths; for example, this is the case if $(\mathcal{F}_t)_{t\geq 0}$ is a Brownian filtration. Let X be an adapted martingale and let $X^* = \sup_{t\geq 0} X_t$, $|X|^* = \sup_{t\geq 0} |X_t|$ denote the associated one- and two-sided maximal functions. In what follows, $\langle X \rangle$ will stand for the corresponding (skew) square bracket: see Dellacherie and Meyer [14] for the definition and basic properties of this object. Next, suppose that Y is the stochastic integral, with respect to X, of some predictable process H which takes values in $[-1, 1]$:

$$Y_t = H_0 X_0 + \int_{0+}^{t} H_s \mathrm{d}X_s.$$

The question about the comparison of the sizes of X and Y has gathered a lot of interest in the literature. See e.g. [10, 11, 12, 13] and consult the monograph [26]. In addition, such stochastic inequalities have found numerous applications in harmonic analysis, where they can be used, among other things, in the study of L^p boundedness of wide classes of Fourier multipliers [4, 5, 6]. We have the following celebrated result proved in [11].

Theorem 1.1. *If X, Y are as above, then for each $1 < p < \infty$ we have*

$$\|Y\|_{L^p} \leq (p^* - 1)\|X\|_{L^p}, \tag{1.1}$$

where $p^* = \max\{p, p/(p-1)\}$. For each p, the constant is the best possible.

There is a powerful method, invented by Burkholder, which allows the efficient study of general class of inequalities for martingales and their stochastic integrals. Roughly speaking, the approach enables to deduce the desired estimate from the existence of a certain special function enjoying appropriate concavity and size requirements. This method (also referred to as the Bellman function method) originates in the theory of optimal control,

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and has turned out to work also in much wider settings of harmonic analysis. See e.g. [8, 24, 25, 26].

For example, in order to prove the above sharp L^p estimate, Burkholder showed that it is enough to find a continuous function $B:\mathbb{R}^2\to\mathbb{R}$ satisfying

1° $B(x,y)\leq 0$ if $|y|\leq |x|$;
2° $B(x,y)\geq |y|^p-(p^*-1)^p|x|^p$

and the following concavity type condition:

3° for any x, y and any h, k with $|k|\leq |h|$, the function $t\mapsto B(x+th,y+tk)$ is concave on $\mathbb{R}$.

See [10, 11, 13] or Chapter 4 in [26] for the relation of such a function to (1.1), consult also Section 2 below. To complete the proof of the L^p bound, Burkholder provides the explicit formula for B:

$$B(x,y)=\alpha_p(|y|-(p^*-1)|x|)(|x|+|y|)^{p-1},$$

where α_p is a certain constant depending only on p. It turns out that this function can be applied in seemingly unrelated areas of mathematics. Namely, there is a deep and unexpected connection of B with the geometric function theory, particularly with the theory of quasiconformal mappings, rank-one convex functionals and the properties of Beurling-Ahlfors operator: see [1, 2, 3, 17, 18] and consult the references therein. In other words, although the function B originates in the probabilistic estimate (1.1), its explicit formula is of independent interest and importance in contexts far and beyond martingale theory.

The above observation was one of the motivations for our research. There is an interesting question concerning the explicit formula for a *weighted* version of Burkholder's function B. Suppose that $W=(W_t)_{t\geq 0}$ is a weight, i.e., a nonnegative and uniformly integrable martingale. It is a usual convention to identify W with its terminal variable W_∞. For any $1\leq p<\infty$ and any weight W, we introduce the associated L^p space as the class of all variables f for which $\|f\|_{L^p(W)}=\left(\int_\Omega |f|^pW\mathrm{d}\mathbb{P}\right)^{1/p}<\infty$. Given a martingale X as above, we will also use the notation $\|X\|_{L^p(W)}=\sup_{t\geq 0}\|X_t\|_{L^p(W)}$. Following Izumisawa and Kazamaki [19], we say that W is an A_p weight (where $1<p<\infty$ is a fixed parameter), if the A_p characteristic of W, given by the formula

$$[W]_{A_p}=\sup_{t\geq 0}\left\|W_t\mathbb{E}(W^{-1/(p-1)}|\mathcal{F}_t)\right\|_\infty,$$

is finite. This is the probabilistic counterpart of the classical, analytic A_p condition introduced by Muckenhoupt [23] during the study of boundedness of the Hardy-Littlewood maximal operator on weighted spaces.

With all the necessary definitions at hand, we can ask about the weighted analogue of (1.1). Namely, for a given and fixed $1<p<\infty$ and a weight W, does there exist a constant $C_{p,W}$ such that we have

$$\|Y\|_{L^p(W)}\leq C_{p,W}\|X\|_{L^p(W)}$$

for all martingales X, Y such that Y is the stochastic integral of X? It can be shown (cf. Domelevo and Petermichl [15], Petermichl and Volberg [28], Wittwer [31]) that the answer is positive if and only if $W\in A_p$. Furthermore, one can show the following optimal factorization of the constant: we have $C_{p,W}\leq c_p[W]_{A_p}^{\max\{1,1/(p-1)\}}$, where c_p depends only on p and the exponent $\max\{1,1/(p-1)\}$ is the best possible. Such extraction of the optimal dependence of the constant on the weight characteristic has gained a lot of interest in the recent literature. For most classical operators in harmonic analysis such an extraction has been carried out successfully: see e.g. [9, 20, 21, 22] and consult the references therein.

Coming back to the context of martingale transforms, the above discussion shows that

$$\|Y\|_{L^p(W)} \leq c_p[W]_{A_p}^{\max\{1,1/(p-1)\}}\|X\|_{L^p(W)}, \qquad 1 < p < \infty. \tag{1.2}$$

Straightforward extrapolation techniques (see e.g. [16] or, in the above probabilistic context, [7]) show that it is enough to study the above bound for the case $p = 2$ only:

$$\|Y\|_{L^2(W)} \leq c_2[W]_{A_2}\|X\|_{L^2(W)}. \tag{1.3}$$

This reduction was used in [15, 28, 31] to establish the above weighted L^p bound. We want to emphasize here that due to this fact, we only construct the Burkholder's function associated with the weighted L^2 estimate (as stated in the abstract) and not for the weighted L^p estimate. To show (1.3), a duality and a number of complicated Bellman functions (involving six variables) were applied. There is a natural question whether the L^2 bound (1.3) can be established directly, in the spirit of Burkholder's approach described earlier. The presence of A_2 weights forces the introduction of two additional arguments and hence the problem reduces to the construction of an explicit function of *four* variables, enjoying appropriate concavity and size conditions similar to 1°-3° above (see Section 2 below for details). The main result of this paper is to give a positive answer to this question. Interestingly, as an immediate by-product, this special function will allow us to obtain a stronger, maximal estimate stated below.

Theorem 1.2. *Suppose that W is an A_p weight, X is a martingale and Y is a stochastic integral, with respect to X, of some predictable process X taking values in $[-1, 1]$. Then for any $1 < p < \infty$ there is a finite constant C_p depending only on p such that*

$$\| \, |Y|^* \|_{L^p(W)} \leq C_p[W]_{A_p}^{\max\{1/(p-1),1\}}\|X\|_{L^p(W)}. \tag{1.4}$$

The exponent $\max\{1/(p-1), 1\}$ *is the best possible.*

As we will see, the function B we provide has quite a complicated structure (which should be compared to its trivial unweighted counterpart: $B(x, y) = y^2 - x^2$). Of course, this increased difficulty is not surprising: in the light of the extrapolation method mentioned above, the weighted L^2 bound implies the validity of (1.2) and hence the corresponding Burkholder's function carries all the information about all L^p estimates in the weighted context. We would like to finish the discussion with a terminological remark. Namely, the function B constructed in this paper yields the constant c_2 in (1.3) which is not optimal. Therefore, in the language used in the Bellman function theory, one could call B a *supersolution* corresponding to (1.3).

The remaining part of the paper is split into two sections. In Section 2 we explain the relation between Burkholder's function and the validity of (1.3). Section 3 contains the explicit construction of the special function and the proof of (1.4).

2. BURKHOLDER'S METHOD

Let us start with the following useful interpretation of A_p weights, valid for $1 < p < \infty$. Fix such a weight W and suppose that $c \geq [W]_{A_p}$. In particular, the finiteness of the A_p characteristic implies the integrability of the function $W^{1/(1-p)}$ and we may consider the associated martingale $V = (V_t)_{t\geq 0}$ given by $V_t = \mathbb{E}(W^{1/(1-p)}|\mathcal{F}_t)$, $t \geq 0$. Note that

Jensen's inequality implies $W_t V_t^{p-1} \geq 1$ almost surely for any $t \geq 0$ and, in addition, the A_p condition is equivalent to the reverse bound

$$W_t V_t^{p-1} \leq [W]_{A_p} \qquad \text{with probability } 1.$$

In other words, an A_p weight of characteristic equal to c gives rise to a two-dimensional martingale (W, V) taking values in the domain

$$\mathcal{D}_c = \{(w, v) \in (0, \infty) \times (0, \infty) \,:\, 1 \leq wv^{p-1} \leq c\}.$$

Note that this martingale terminates at the lower boundary of this domain: $W_\infty V_\infty^{p-1} = 1$ almost surely. Actually, the implication can be reversed. Given a pair (W, V) taking values in $\mathcal{D}_c$ and terminating at the set $wv^{p-1} = 1$, one easily checks that its first coordinate is an A_p weight with $[W]_{A_p} \leq c$.

Let $c \geq 1$ be a fixed parameter. Suppose that $G : \mathbb{R}^2 \times \mathcal{D}_c \to \mathbb{R}$ is a given Borel function and assume that we are interested in showing that

$$\mathbb{E}G(X_t, Y_t, W_t, V_t) \leq 0, \qquad t \geq 0. \tag{2.1}$$

Here (X, Y) is an arbitrary pair of martingales such that Y is the stochastic integral, with respect to X, of some predictable process with values in $[-1, 1]$, and (W, V) is a pair associated with some A_p weight of characteristic not bigger than c. A key to handle this problem is to consider a C^2 function $B : \mathbb{R}^2 \times \mathcal{D}_c \to \mathbb{R}$ which satisfies the following properties:

1° (Initial condition) We have $B(x, y, w, v) \leq 0$ if $|y| \leq |x|$ and $1 \leq \mathtt{wv}^{p-1} \leq c$.

2° (Majorization property) We have $B \geq G$ on $\mathbb{R}^2 \times \mathcal{D}_c$.

3° (Concavity-type property) For any $(x, y, w, v) \in \mathbb{R}^2 \times \mathcal{D}_c$ and $d, e, r, s \in \mathbb{R}$ satisfying $|e| \leq |d|$, the function

$$\xi_B(t) := B(x + td, y + te, w + tr, v + ts),$$

given for those t, for which $1 \leq (w + tr)(v + ts) \leq c$, is locally concave.

The connection between the existence of such a function and the validity of (2.1) is described in the lemma below.

Lemma 2.1. *Let $1 < p < \infty$ and $c \geq 1$ be fixed. If B satisfies the conditions 1°, 2° and 3°, then the inequality (2.1) holds true for all X, Y, W and V as above.*

Proof. The argument rests on Itô's formula. Consider an auxiliary process $Z = (X, Y, W, V)$. Since B is of class C^2, we may write

$$B(Z_t) = I_0 + I_1 + I_2/2,$$

where

$$\begin{aligned} I_0 &= B(Z_0), \\ I_1 &= \int_{0+}^t B_x(Z_u)\mathrm{d}X_u + \int_{0+}^t B_y(Z_u)\mathrm{d}Y_u + \int_{0+}^t B_w(Z_u)\mathrm{d}W_u + \int_{0+}^t B_v(Z_u)\mathrm{d}V_u, \\ I_2 &= \int_{0+}^t D^2 B(Z_u)\mathrm{d}\langle Z\rangle_u. \end{aligned}$$

Here $D^2 B$ is the Hessian matrix of B and in the definition of I_2 we have used a shortened notation for the sum of all second-order terms. Let us study the properties of the terms I_0, I_1 and I_2. The first of them is nonpositive because of the condition 1°. The expectation of I_1 is zero, by the properties of stochastic integrals. To handle the last term, note that by a simple differentiation, 3° implies

$$\left\langle D^2B(x,y,w,v)(d,e,r,s),(d,e,r,s)\right\rangle \leq 0$$

for any $(x,y,w,v) \in \mathbb{R}^2 \times \mathcal{D}_c$ and any $(d,e,r,s) \in \mathbb{R}^4$ satisfying $|e| \leq |d|$. This implies $I_2 \leq 0$, by a straightforward approximation of the integral by Riemann sums. Putting all the above observations together, we get $\mathbb{E}B(Z_t) \leq 0$, which combined with the majorization condition 2° gives the assertion. □

We conclude this section with three observations.

Remark 2.1. The above statement is true without the assumption that B is of class C^2: it is enough to ensure that B is continuous. Indeed, the condition 3° guarantees that any possible 'cusp' of B is of concave type and hence the argument works. More precisely, this can be proved by standard mollification argument (consult e.g. Domelevo and Petermichl [15] or Wang [30]). There are also several other methods of showing this. One can use the appropriate extension of Itô's formula developed in [27]; alternatively, one can first establish the corresponding estimate for (discrete-time) martingales and use approximation: see [10] for details.

Remark 2.2. The above approach works also in the unweighted case, which corresponds to the choice $c = 1$. Then the processes W and V are constant, and hence the special function B depends only on the variables x, y. This brings us back to the original setting considered by Burkholder.

Remark 2.3. The above approach is very flexible and can be easily modified to other contexts. For example, suppose that we are interested in the maximal bound of the form

$$\mathbb{E}G(X_t,Y_t,Y_t^*,W_t,V_t) \leq 0, \qquad t \geq 0,$$

for all X, Y, W and V as in (2.1). Here $Y_t^* = \max_{0\leq s\leq t} Y_t$ is the truncated one-sided maximal function of Y. Then it is enough to construct $B : \{(x,y,z,w,v) \in \mathbb{R}^3 \times \mathcal{D}_c : y \leq z\} \to \mathbb{R}$ satisfying

1° (Initial condition) We have $B(x,y,y,w,v) \leq 0$ if $|y| \leq |x|$ and $1 \leq \mathtt{w}\mathtt{v}^{p-1} \leq c$.
2° (Majorization property) We have $B \geq G$.
3° (Concavity-type property) For any $(x,y,z,w,v) \in \mathbb{R}^3 \times \mathcal{D}_c$ and $d,e,r,s \in \mathbb{R}$ satisfying $y < z$ and $|e| \leq |d|$, the function

$$\xi_B(t) := B(x+td, y+te, z, w+tr, v+ts),$$

given for those t, for which $1 \leq (w+tr)(v+ts) \leq c$, is locally concave. Furthermore, we have $B_z(x,y,y,w,v) \leq 0$.

Again, the proof rests on Itô's formula. The additional requirement formulated at the end of 3° enables us to handle the additional stochastic integral $\int_{0+}^t B_z(X_s,Y_s,Y_s^*,W_s,V_s)\mathrm{d}X_s^*$ (and guarantees that this integral is nonpositive).

3. A SPECIAL FUNCTION

Throughout this section, $c > 1$ is a fixed parameter (which corresponds to the 'truly' weighted context). Again, as discussed in the paragraph following (1.3), we only consider the case $p = 2$. The main result of this section is the following.

Theorem 3.1. *There is a continuous function $B : \mathbb{R}^2 \times \mathcal{D}_c \to \mathbb{R}$ satisfying 1°-3° with $G(x,y,w,v) = \kappa(y^2w - C^2c^2x^2v^{-1})$ for some positive universal constants κ and C.*

The above statement combined with Lemma 2.1 yields the validity of (1.3), by passing $t \to \infty$ and using standard limiting arguments. A slightly stronger, maximal estimate announced in Introduction will be proved at the end of this section. Assume the D^1, D^2, D^3 are the 'angular' subsets of $\mathbb{R}^2 \times \mathcal{D}_c$ given by

$$\begin{aligned} D^1 &= \left\{(x,y,w,v) : |y| \geq 20c|x|(c/t)^{1-\beta}\right\}, \\ D^2 &= \left\{(x,y,w,v) : 10|x| \leq |y| \leq 20c|x|(c/t)^{1-\beta}\right\}, \\ D^3 &= \{(x,y,w,v) : |y| \leq 10|x|\}. \end{aligned} \tag{3.1}$$

Here and in what follows, we denote $t = wv$. Define the functions $b_i : \mathcal{D}_c \to \mathbb{R}$ by

$$\begin{aligned} b_1(x,y,w,v) &= y^2 w\phi(wv), \\ b_2(x,y,w,v) &= y^2(2v)^{-1}, \\ b_3(x,y,w,v) &= c^2x^2v^{-1}, \\ b_4(x,y,w,v) &= c^\beta|x||y|w^{1-\beta}v^{-\beta}, \\ b_5(x,y,w,v) &= c^\beta y^2 w^{1-\beta}v^{-\beta}, \\ b_6(x,y,w,v) &= c^2x^2w\psi(wv), \end{aligned}$$

where $\beta = 3/4$ and ϕ, ψ are functions from $[1,c]$ to $\mathbb{R}$ given by

$$\phi(t) = 2 - \frac{1}{t} - \frac{\ln(t)}{2c}, \qquad \psi(t) = (t\phi(t))^{-1}.$$

Furthermore, set $U(x,y,w,v) = b_1 - b_2 - 320000b_3 - 294400b_6$. Now we are finally ready to introduce the explicit formula for the desired Burkholder's function B:

$$B(x,y,w,v) = \begin{cases} B_1(x,y,w,v) & \text{on } D^1, \\ B_2(x,y,w,v) & \text{on } D^2, \\ B_3(x,y,w,v) & \text{on } D^3, \end{cases}$$

where B_1, B_2, $B_3 : \mathbb{R}^2 \times \mathcal{D}_c \to \mathbb{R}$ are given by

$$\begin{aligned} B_1(x,y,w,v) &= U(x,y,w,v) + 6400c^2x^2v^{-1}, \\ B_2(x,y,w,v) &= U(x,y,w,v) + 320b_4(x,y,w,v), \\ B_3(x,y,w,v) &= U(x,y,w,v) + 32b_5(x,y,w,v). \end{aligned}$$

As we have already announced earlier, this function has quite a complicated form, it is actually defined with three different formulas on three separate domains. Let us verify that it satisfies the conditions 1°-3° listed in the formulation of Lemma 2.1. The first two properties are relatively easy to prove; the main difficulty lies in establishing the concavity condition. We start with the easy part. Note that the second half of the lemma below implies the continuity of B.

Lemma 3.1. *The function B satisfies the properties 1° and 2°. Furthermore, we have $B_1 \leq B_2$ on D^1, $B_2 \leq \min(B_1, B_3)$ on D^2 and $B_3 \leq B_2$ on D^3.*

Proof. To check the initial condition, note that for $|y| \leq |x|$ we have $(x,y,w,v) \in D^3$. Furthermore, from $\phi(t) \leq 2$, we obtain

$$\begin{aligned} B_3(x,y,w,v) \leq b_1 + 32b_5 - 320000b_3 &\leq x^2w(2 + 32c^\beta(wv)^{-\beta} - 320000(c^2/t)) \\ &\leq x^2w(2 + 32(c/t)^\beta - 320000(c/t)c) \leq 0. \end{aligned}$$

Let us now study the majorization. Observe that

$$b_1 - b_2 = y^2 w \left[2 - \frac{1}{t} - \frac{\ln t}{2c} - \frac{1}{2t}\right] \geq \frac{1}{2} y^2 w,$$

because the function in the square bracket is increasing and has its minimum at the point $t = 1$. Now from the estimate $\phi(t) \geq 1$ we have that $\psi(t) \leq 1/t$ and as a consequence,

$$320000 b_3 + 294400 b_6 \leq c^2 x^2 w \left(320000 \frac{1}{t} + 294400 \frac{1}{t}\right) = 614400 c^2 x^2 v^{-1}.$$

Finally, we have

$$\begin{aligned} B \geq b_1 - b_2 - 320000 b_3 - 294400 b_6 &\geq \frac{1}{2} y^2 w - 614400 c^2 x^2 v^{-1} \\ &= \frac{1}{2}(y^2 w - 1228800 c^2 x^2 v^{-1}), \end{aligned}$$

so the condition 2° is satisfied with $\kappa = 1/2$ and $C = (1228800)^{1/2} < 1109$.

It remains to verify the relations between B_1, B_2 and B_3. If $(x, y, w, v) \in D^1$, then

$$320 b_4 = 320 c^\beta |x||y| w^{1-\beta} v^{-\beta} \geq 6400 c^2 x^2 v^{-1},$$

so $B_2 \geq B_1$. If $(x, y, w, v) \in D^2$, we have reverse inequality $B_2 \leq B_1$. Furthermore, on D^2 we have

$$320 b_4 = 320 c^\beta |x||y| w^{1-\beta} v^{-\beta} \leq 32 c^\beta |y|^2 w^{1-\beta} v^{-\beta} = 32 b_5,$$

which is exactly $B_2 \leq B_3$. To finish the proof, observe that the above estimate is reversed on D^3. □

We turn our attention to the crucial condition 3°. From symmetry (and the equality $B_x(0, y, w, v) = 0$ for all y, w, v), without loss of generality, we may only consider points $(x, y, w, v) \in \mathbb{R}^2 \times \mathcal{D}_c$ such that x and y are nonnegative. Furthermore, it is enough to verify the version of the concavity "localized" to each D^i. More precisely, it suffices to show that for each i, each $(x, y, w, v) \in D^i$ and any $d, e, r, s \in \mathbb{R}$ satisfying $|e| \leq |d|$, the function

$$\xi_{B_i}(t) := B_i(x + td, y + te, w + tr, v + ts),$$

given for those t, for which $(x+td, t+te, w+tr, v+ts) \in D^i$, satisfies $\xi''_{B_i}(0) \leq 0$. To see that this is sufficient, suppose that we have successfully established the localized concavity and pick an arbitrary point (x, y, w, v) from the domain of B. By the continuity of B, we may and do assume that x and y are not both 0. If (x, y, w, v) belongs to the interior of some D^i, then $\xi_B(t) = \xi_{B^i}(t)$ for t sufficiently close to 0 and hence $\xi''_B(0) = \xi''_{B_i}(0) \leq 0$. On the other hand, if (x, y, w, v) lies on the common boundary of two sets D^i and D^j, then by the second part of the above lemma,

$$\xi_B(t) = \min\{\xi_{B_i}(t), \xi_{B_j}(t)\} = \begin{cases} \xi_{B^i}(t) & \text{if } (x + td, t + te, w + tr, v + ts) \in D^i, \\ \xi_{B^j}(t) & \text{if } (x + td, t + te, w + tr, v + ts) \in D^j \end{cases}$$

for t sufficiently close to 0. Hence, if ξ_B had a convex "cusp" at zero, then the same would be true for ξ_{B^i} and ξ_{B^j}, which contradicts the localized concavity. This establishes the desired property 3°.

The localized concavity will be accomplished by a careful analysis of the derivatives $\xi''_{b_j}(0)$ of the building blocks b_j, $j = 1, 2, \ldots, 6$. In the next lemma we gather estimates for the parts b_1 and b_6.

Lemma 3.2. *We have the following estimates on the quadratic forms associated with the functions b_1 and b_6:*

(a) $\xi''_{b_1}(0) \le 80cwe^2$,
(b) $\xi''_{b_1}(0) \le 4we^2 + 8y|e||r|$,
(c) $\xi''_{b_6}(0) \ge (1/16)cv^{-3}x^2s^2$.

Proof. (a) It is equivalent to showing the nonpositive-definiteness of the matrix

$$\mathbb{A}(y,w,v) = \begin{pmatrix} 2w\phi(t) - 80cw & 2y\phi(t) + 2yt\phi'(t) & 2yw^2\phi'(t) \\ 2y\phi(t) + 2yt\phi'(t) & 2y^2v\phi'(t) + y^2tv\phi''(t) & 2y^2w\phi'(t) + y^2tw\phi''(t) \\ 2yw^2\phi'(t) & 2y^2w\phi'(t) + y^2tw\phi''(t) & y^2w^3\phi''(t) \end{pmatrix}.$$

From Sylvester's criterion, it is enough to prove that

$$y^2w^3\phi''(t) \le 0, \tag{3.2}$$

$$\det \begin{pmatrix} 2y^2v\phi'(t) + y^2tv\phi''(t) & 2y^2w\phi'(t) + y^2tw\phi''(t) \\ 2y^2w\phi'(t) + y^2tw\phi''(t) & y^2w^3\phi''(t) \end{pmatrix} \ge 0 \tag{3.3}$$

and

$$\det \mathbb{A}(y,w,v) \le 0. \tag{3.4}$$

The inequality (3.2) follows immediately from $t \in [1,c]$ and the estimate

$$y^2w^3\phi''(t) = -\frac{y^2w^3}{2ct^3}(4c - t) \le 0.$$

The inequality (3.3) is equivalent to $\phi'(t)(2\phi'(t) + t\phi''(t)) \le 0$ and follows from

$$\phi'(t) = \frac{1}{2ct^2}(2c - t) \ge 0 \qquad \text{and} \qquad 2\phi'(t) + t\phi''(t) = -\frac{1}{2ct} \le 0.$$

In order to show (3.4) we simplify the matrix $\mathbb{A}$ by carrying out some elementary operations. The determinant of $\mathbb{A}$ has the same sign as

$$\det \begin{pmatrix} -80cw & 2\phi(t) + 2t\phi'(t) & 0 \\ 2\phi(t) & 0 & 2\phi'(t) \\ 2w\phi'(t) & 2\phi'(t) + t\phi''(t) & w\phi''(t) \end{pmatrix}$$
$$= 4w\left[\left(2(\phi'(t))^2 - \phi(t)\phi''(t)\right)\left(\phi(t) + t\phi'(t)\right) + 40c\phi'(t)(2\phi'(t) + t\phi''(t))\right].$$

We compute that

$$\phi(t) + t\phi'(t) = 2 - \frac{\ln(t)}{2c} - \frac{1}{2c} \le 2, \tag{3.5}$$

$$2(\phi'(t))^2 = \phi'(t)\frac{2c - t}{ct^2} \le \frac{2\phi'(t)}{t}$$

and, since $\phi(t) \le 2$,

$$-\frac{\phi(t)\phi''(t)}{\phi'(t)} \le \frac{2\left(\frac{2}{t^3} - \frac{1}{2ct^2}\right)}{\frac{1}{t^2} - \frac{1}{2ct}} \le \frac{8}{t}. \tag{3.6}$$

Combining these facts we obtain

$$(2(\phi'(t))^2 - \phi(t)\phi''(t))(\phi(t) + t\phi'(t)) \le \frac{20\phi'(t)}{t},$$

and since

$$40c\phi'(t)(2\phi'(t)+t\phi''(t)) = -\frac{20\phi'(t)}{t},$$

the inequality (3.4) is satisfied. This completes the proof of the part (a).

(b) Firstly, observe that it is sufficient to prove the nonpositive-definiteness of the matrix

$$\mathbb{B}(y,w,v) = \begin{pmatrix} 2w\phi(t)-4w & 0 & 2yw^2\phi'(t) \\ 0 & 2y^2v\phi'(t)+y^2tv\phi''(t) & 2y^2w\phi'(t)+y^2tw\phi''(t) \\ 2yw^2\phi'(t) & 2y^2w\phi'(t)+y^2tw\phi''(t) & y^2w^3\phi''(t) \end{pmatrix}.$$

Indeed, we have the estimate

$$\begin{aligned} \xi''_{b_1}(0) &= \langle \mathbb{B}(y,w,v)(e,r,s),(e,r,s)\rangle + 4we^2 + 2(2y\phi(t)+2yt\phi'(t))er \\ &\le 4we^2 + 4(y\phi(t)+yt\phi'(t))|e||r| \le 4we^2 + 8y|e||r|, \end{aligned}$$

where the last inequality follows from (3.5).

From Sylvester's criterion, the nonpositive-definiteness of the matrix $\mathbb{B}$ is equivalent to inequalities (3.2) and (3.3) (which we already showed in the proof of the (a) part of the lemma) and the estimate

$$\det \mathbb{B}(y,w,v) \le 0. \tag{3.7}$$

By carrying out some elementary operations we show that the determinant of $\mathbb{B}$ has the same sign as

$$\begin{aligned} &\det \begin{pmatrix} 2w\phi(t)-4w+2wt\phi'(t) & 0 & 2w\phi'(t) \\ 0 & 0 & 2\phi'(t) \\ 2w\phi'(t)+wt\phi''(t) & 2\phi'(t)+t\phi''(t) & w\phi''(t) \end{pmatrix} \\ &= -(2\phi'(t)+t\phi''(t))2\phi'(t)(2w\phi(t)-4w+2wt\phi'(t)). \end{aligned}$$

However, we compute that

$$2\phi'(t)+t\phi''(t) = -\frac{1}{2tc} \le 0.$$

So, since $\phi(t)+t\phi'(t)\le 2$ and $\phi'(t)\ge 0$, the inequality (3.7) is satisfied.

(c) In analogy to the above considerations, we must show that the matrix

$$\begin{aligned} &\mathcal{C}(x,w,v) \\ &= \begin{pmatrix} 2c^2w\psi(t) & 2xc^2(\psi(t)+t\psi'(t)) & 2xc^2w^2\psi'(t) \\ 2xc^2(\psi+t\psi'(t)) & x^2c^2(2v\psi'(t)+wv^2\psi''(t)) & x^2c^2(2w\psi'(t)+w^2v\psi''(t)) \\ 2xc^2w^2\psi'(t) & x^2c^2(2w\psi'(t)+w^2v\psi''(t)) & x^2c^2w^3\psi''(t)-\frac{1}{16}cv^{-3}x^2 \end{pmatrix} \end{aligned}$$

is nonpositive-definite. For notational convenience, let us define the function $\widehat{\psi}:[1,c]\longrightarrow \mathbb{R}$ as $\widehat{\psi}(t)=t\psi(t)=(\phi(t))^{-1}$ and set

$$d(t,x) = x^2c^2(2v^{-3}\widehat{\psi} - 2v^{-2}w\widehat{\psi}' + v^{-1}w^2\widehat{\psi}'') - \frac{1}{16}v^{-3}cx^2.$$

Then we can rewrite the matrix $\mathcal{C}$ as

$$\begin{pmatrix} 2c^2\widehat{\psi}(t)v^{-1} & 2c^2x\widehat{\psi}'(t) & 2xc^2(\widehat{\psi}'(t)wv^{-1}-\widehat{\psi}(t)v^{-2}) \\ 2c^2x\widehat{\psi}'(t) & c^2vx^2\widehat{\psi}''(t) & c^2wx^2\widehat{\psi}''(t) \\ 2xc^2(\widehat{\psi}'(t)wv^{-1}-\widehat{\psi}(t)v^{-2}) & c^2wx^2\widehat{\psi}''(t) & d(x,t) \end{pmatrix}.$$

Again, from Sylvester's criterion, we reduce the problem to checking the signs of appropriate minors. More precisely, we will show that

$$2c^2\widehat{\psi}v^{-1} \geq 0, \tag{3.8}$$

$$\det\begin{pmatrix} 2c^2\widehat{\psi}(t)v^{-1} & 2c^2x\widehat{\psi}'(t) \\ 2c^2x\widehat{\psi}'(t) & c^2vx^2\widehat{\psi}''(t)\end{pmatrix} \geq 0 \tag{3.9}$$

and

$$\det\mathcal{C}(x,w,v) \geq 0. \tag{3.10}$$

The inequality (3.8) is obvious. Condition (3.9) is equivalent to $\widehat{\psi}(t)\widehat{\psi}''(t)-2(\widehat{\psi}'(t))^2 \geq 0$, which is a consequence of the definition $\widehat{\psi}(t) = (\phi(t))^{-1}$ and the inequality $\phi''(t) \leq 0$. To show (3.10), we perform certain elementary operations on the columns and rows of the matrix to prove that the determinant of $\mathcal{C}$ has the same sign as

$$\det\begin{pmatrix} 2\widehat{\psi}(t)v^{-1} & 2\widehat{\psi}'(t) & 0 \\ 2\widehat{\psi}'(t) & v\widehat{\psi}''(t) & 2v^{-1}\widehat{\psi}'(t) \\ -2\widehat{\psi}(t)v^{-2} & 0 & -2wv^{-2}\widehat{\psi}'(t) - \frac{1}{16}v^{-3}c^{-1}\end{pmatrix}$$
$$= 2v^{-3}\left(\left(-2\widehat{\psi}'(t)t - \frac{1}{16}c^{-1}\right)(\widehat{\psi}(t)\widehat{\psi}''(t) - 2(\widehat{\psi}'(t))^2) - 4\widehat{\psi}(t)(\widehat{\psi}'(t))^2\right).$$

We compute that

$$\widehat{\psi}'(t) = -\frac{\phi'(t)}{\phi^2(t)},$$

$$\widehat{\psi}''(t) = -\frac{\phi(t)\phi''(t) - 2(\phi'(t))^2}{\phi^3(t)}$$

and

$$\widehat{\psi}(t)\widehat{\psi}''(t) - 2(\widehat{\psi}'(t))^2 = -\frac{\phi''(t)}{\phi^3(t)}.$$

Hence we need to show that

$$-2\phi'(t)\phi''(t)t + \frac{1}{16}c^{-1}\phi''(t)(\phi(t))^2 - 4(\phi'(t))^2 \geq 0.$$

Now observe that

$$-2\phi'(t)\phi''(t)t - 4(\phi'(t))^2 = -2\phi'(t)(2\phi'(t) + \phi''(t)t) = \phi'(t)c^{-1}t^{-1}$$

and from (3.6) and $\phi(t) \leq 2$

$$\frac{1}{16}c^{-1}\phi''(t)(\phi(t))^2 \geq \frac{-\phi(t)\phi'(t)}{2ct} \geq -\frac{\phi'(t)}{ct},$$

which completes the proof of the lemma. □

In the series of three lemmas below we will show that the function B satisfies required concavity condition. Let us start with the domain D^1.

Lemma 3.3. *We have $\xi''_{b_1-b_2-160b_3}(0) \leq 0$ for any $(x,y,w,v) \in D^1$ and (d,e,r,s) such that $|e| \leq |d|$.*

Remark 3.2. This lemma handles the property 3° on the domain D^1. Indeed, the additional summands $-319840b_3$ and $-294400b_6$ are concave functions (concavity of $-b_3$ is easy to check, concavity of $-b_6$ follows from part (c) of Lemma 3.2).

Proof of Lemma 3.3. We have that

$$\xi''_{b_2}(0) = \frac{1}{v}\left(e - \frac{ys}{v}\right)^2, \qquad \xi''_{b_3}(0) = \frac{2c^2}{v}\left(d - \frac{xs}{v}\right)^2.$$

Now consider two cases. If $|d - \frac{xs}{v}| \geq d/2$, then from above formulas and part (a) of Lemma 3.2 we obtain

$$\begin{aligned}\xi''_{b_1-b_2-160b_3}(0) &\leq 80cwe^2 - \frac{1}{v}\left(e - \frac{ys}{v}\right)^2 - \frac{320c^2}{v}\left(d - \frac{xs}{v}\right)^2 \\ &\leq 80cwe^2 - 80c^2v^{-1}d^2 \\ &\leq 80cwd^2 - 80cwd^2 \\ &= 0.\end{aligned}$$

If $|d - \frac{xs}{v}| < d/2$, then

$$\begin{aligned}\frac{ys}{dv} - \frac{e}{d} &= \frac{ys}{dv} - \frac{y}{x} + \frac{y}{x} - \frac{e}{d} = y\left(\frac{s}{dv} - \frac{1}{x}\right) + \frac{y}{x} - \frac{e}{d} = \frac{y}{x}\left(\frac{sx}{vd} - 1 + 1 - \frac{e}{d}\frac{x}{y}\right) \\ &\geq \frac{y}{x}\left(-\frac{1}{2} + 1 - \frac{1}{20c}\right) \geq 20c\left(\frac{1}{2} - \frac{1}{20c}\right) = 10c - 1 \geq 9c.\end{aligned}$$

Hence

$$\xi''_{b_1-b_2-160b_3}(0) \leq \xi''_{b_1-b_2}(0) \leq 80cwe^2 - \frac{1}{v}d^2 9^2 c^2 \leq 80cwd^2 - 81cwd^2 \leq 0.$$

The proof is complete. □

The next lemma discusses the concavity condition in the middle domain D^2.

Lemma 3.4. *We have $\xi''_{b_1+320b_4-320000b_3}(0) \leq 0$ for any $(x,y,w,v) \in D^2$ and (d,e,r,s) such that $|e| \leq |d|$.*

Remark 3.3. This lemma handles the property 3° on the domain D^2. Indeed, functions $-b_2$ and $-294400b_6$ are concave, so they do not affect the condition 3°.

Proof of Lemma 3.4. Let $D = \frac{d}{x}$, $E = \frac{e}{y}$, $R = \frac{r}{w}$ and $S = \frac{s}{v}$. We compute that

$$\begin{aligned}\xi_{b_4-1000b_3}(0) &= c^\beta xyw^{1-\beta}v^{-\beta}\langle A_1(E,D,R,S),(E,D,R,S)\rangle \\ &\quad - 1000c^2x^2v^{-1}\langle A_2(D,S),(D,S)\rangle,\end{aligned}$$

where the matrices A_1 and A_2 are defined as

$$A_1 = \begin{pmatrix} 0 & 1 & 1-\beta & -\beta \\ 1 & 0 & 1-\beta & -\beta \\ 1-\beta & 1-\beta & \beta(\beta-1) & \beta(\beta-1) \\ -\beta & -\beta & \beta(\beta-1) & \beta(\beta+1)\end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2\end{pmatrix}.$$

From the assumption $y \geq 10x$ and differential subordination ($|e| \leq |d|$) we obtain

$$|E| = \left|\frac{e}{y}\right| \leq \frac{1}{10}\left|\frac{d}{x}\right| = \frac{1}{10}|D|,$$

so $E = \lambda D$, where λ is a constant with absolute value bounded by $\frac{1}{10}$. So, we can reduce Hessians to three variables (D,R and S): we have

$$\begin{aligned}\xi''_{b_4-1000b_3}(0) = c^\beta xyw^{1-\beta}v^{-\beta}\left\langle A_3(D,R,S),(D,R,S)\right\rangle \\ - 1000c^2x^2v^{-1}\left\langle A_4(D,R,S),(D,R,S)\right\rangle,\end{aligned} \tag{3.11}$$

where matrices A_3 and A_4 are defined as

$$A_3 = \begin{pmatrix} 2\lambda & (1-\beta)(1+\lambda) & -\beta(1+\lambda) \\ (1-\beta)(1+\lambda) & \beta(\beta-1) & \beta(\beta-1) \\ -\beta(1+\lambda) & \beta(\beta-1) & \beta(\beta+1) \end{pmatrix}$$

and

$$A_4 = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{pmatrix}.$$

Now from $y \leq 20cx(c/t)^{1-\beta}$ we have the estimate

$$1000c^2x^2v^{-1} \geq 50xyv^{-1}c(c/t)^{\beta-1} = 50c^\beta xyw^{1-\beta}v^{-\beta}.$$

Obviously, A_4 is nonnegative-definite. Hence, from the above inequality and (3.11), we obtain

$$\xi''_{b_4-1000b_3}(0) \leq c^\beta xyw^{1-\beta}v^{-\beta}\left\langle (A_3-50A_4)(D,R,S),(D,R,S)\right\rangle.$$

It is enough to show that

$$\left\langle (A_3-50A_4)(D,R,S),(D,R,S)\right\rangle \leq -\frac{1}{20}D^2 - \frac{1}{40}|D||R|. \tag{3.12}$$

Indeed, from the above inequalities and part (b) of Lemma 3.2 we have that

$$\begin{aligned}\xi''_{b_1+320(b_4-1000b_3)}(0) &\leq 4we^2 + 8y|e||r| - 16c^\beta xyw^{1-\beta}v^{-\beta}D^2 - 8c^\beta xyw^{1-\beta}v^{-\beta}|D||R| \\ &\leq 4wd^2 + 8y|d||r| - 16wd^2(c/t)^\beta(y/x) - 8y|d||r|(c/t)^\beta \\ &\leq 0.\end{aligned}$$

To finish the proof of the lemma, observe that the estimate (3.12) is equivalent to nonpositive-definiteness of the matrix

$$\begin{pmatrix} 2\lambda - 100 + \frac{1}{20} & \frac{1}{4}(1+\lambda) \pm \frac{1}{40} & -\frac{3}{4}(1+\lambda)+100 \\ \frac{1}{4}(1+\lambda) \pm \frac{1}{40} & -\frac{3}{16} & -\frac{3}{16} \\ -\frac{3}{4}(1+\lambda)+100 & -\frac{3}{16} & \frac{21}{16}-100 \end{pmatrix},$$

for every $|\lambda| \leq 1/10$, which we check by straightforward calculation (determinant of this matrix is convex as a function of λ, so it is sufficient to check only two endpoint cases $\lambda = 1/10$ and $\lambda = -1/10$). □

Finally, we prove the concavity condition in the domain D^3 in the last lemma.

Lemma 3.5. *We have* $\xi''_{b_1+32b_5-4600b_3-294400b_6}(0) \leq 0$ *for any* $(x,y,w,v) \in D^3$ *and* (d,e,r,s) *such that* $|e| \leq |d|$.

Remark 3.4. This lemma handles the property 3° on the domain D^3. Indeed, the additional summands $-b_2$ and $-315400b_3$ are concave, so they do not affect the concavity.

Proof of Lemma 3.5. We use the same notation for relative changes D, E, R and S as in the proof of the previous lemma. We start with the analysis of the part b_3. We have that

$$\xi''_{b_3}(0) = \frac{2c^2}{v}\left(d - \frac{xs}{v}\right)^2 = \frac{2c^2x^2}{v}(D-S)^2 \geq \frac{2cx^2}{v}(D-S)^2 = \frac{2cx^2w}{t}(D-S)^2.$$

From the part (c) of Lemma 3.2 and the above estimate we obtain

$$\begin{aligned}\xi''_{b_3+64b_6}(0) &\geq 2\left(\frac{c}{t}x^2w(D^2-2DS+S^2) + 2cv^{-3}x^2s^2\right)\\ &= 2\left(\frac{c}{t}x^2w(D^2-2DS+S^2) + 2\left(\frac{c}{t}\right)x^2wS^2\right)\\ &= 2\left(\frac{c}{t}\right)x^2w(D^2-2DS+3S^2)\\ &\geq \left(\frac{c}{t}\right)x^2w(D^2+2S^2)\\ &\geq \left(\frac{c}{t}\right)y^2w\left(E^2+\frac{2}{100}S^2\right),\end{aligned}$$

hence

$$\xi''_{16^{-1}\cdot 23\cdot 100(b_3+64b_6)}(0) \geq \left(\frac{c}{t}\right)y^2w16^{-1}[2300E^2+46S^2]. \tag{3.13}$$

Now we turn our attention to the analysis of the part b_5. We have that

$$\xi''_{b_5}(0) = \left(\frac{c}{t}\right)^{\beta} y^2w\left\langle A_5(E,R,S),(E,R,S)\right\rangle,$$

where

$$A_5 = \begin{pmatrix} 2 & 2(1-\beta) & -2\beta\\ 2(1-\beta) & \beta(\beta-1) & \beta(\beta-1)\\ -2\beta & \beta(\beta-1) & \beta(\beta+1)\end{pmatrix}.$$

We check by straightforward calculation that

$$\begin{pmatrix} 2 & 2(1-\beta) & -2\beta\\ 2(1-\beta) & \beta(\beta-1) & \beta(\beta-1)\\ -2\beta & \beta(\beta-1) & \beta(\beta+1)\end{pmatrix} \leq 16^{-1}\begin{pmatrix}192 & \pm 2 & 0\\ \pm 2 & 0 & 0\\ 0 & 0 & 46\end{pmatrix}.$$

So

$$\xi''_{b_5}(0) \leq \left(\frac{c}{t}\right)^{\beta} y^2w16^{-1}(192E^2 - 4|E||R| + 46S^2)$$

and, from (3.13),

$$\begin{aligned}\xi''_{b_5-16^{-1}\cdot 23\cdot 100(b_3+64b_6)}(0) &\leq \left(\frac{c}{t}\right)^{\beta} y^2w16^{-1}(-2108E^2 - 4|E||R|)\\ &\leq y^2w16^{-1}(-2108E^2-4|E||R|).\end{aligned}$$

Now, from the above estimate an part (b) of Lemma 3.2, we have that

$$\xi''_{b_1+32b_5-4600b_3-294400b_6} \leq -4212we^2 \leq 0,$$

which concludes the proof. □

We conclude by proving the maximal inequality formulated in the introductory section.

Proof of (1.4). By the extrapolation argument, it is enough to show the estimate for $p = 2$ only. Fix $c > 1$ and consider the function $\mathbb{B} : \{(x, y, z, w, v) \in \mathbb{R}^3 \times \mathcal{D}_c : x \leq z\} \to \mathbb{R}$ given by $\mathbb{B}(x, y, z, w, v) = B(x, y - z, w, v)$. This new object enjoys the properties listed in Remark 2.3 above (in 2°, it majorizes $\mathbb{G}(x, y, z, w, v) = G(x, y - z, w, v) = \kappa((y - z)^2 w - C^2 c^2 x^2 v^{-1}))$; in particular, we have $\mathbb{B}_z(x, y, y, w, v) = B_y(x, 0, w, v) = 0$, by the symmetry of B. Hence we obtain

$$\mathbb{E}(Y_t^* - Y_t)^2 W_t \leq C^2 c^2 \mathbb{E} X_t^2 V_t^{-1}$$

for any X, Y such that Y is a stochastic integral of X and any pair (W, V) associated with an A_2 weight of characteristic not exceeding c. Letting $t \to \infty$ and using some standard limiting arguments (and the equality $V_\infty^{-1} = W_\infty$), we get

$$\|Y^* - Y\|_{L^2(W)} \leq C[W]_{A_2} \|X\|_{L^2(W)}.$$

It remains to use the fact that the two-sided maximal function $|Y|^*$ satisfies $|Y|^* \leq |Y^*| + |(-Y)^*|$, which implies

$$\||Y|^*\|_{L^2(W)} \leq \|Y^*\|_{L^2(W)} + \|(-Y)^*\|_{L^2(W)} \leq 4C[W]_{A_2} \|X\|_{L^2(W)}.$$

The proof is complete. □

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