BLO CLASS UNDER THE CHANGE OF MEASURE

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ABSTRACT. The purpose of the paper is to study the behavior of the classes of functions of bounded lower oscillation under the change of measure given by the weight w. More specifically, we provide sharp upper and lower bounds for the norm of the inclusion $BLO \hookrightarrow BLO(w)$ in terms of A_∞ constant of the weight. The results hold in the general context of probability spaces equipped with a treelike structure.

1. Introduction

The motivation for the results obtained in this paper comes from a very natural question about structural properties of some classical spaces of harmonic analysis. We start from introducing the basic background and notation. Let f be a real-valued locally integrable function on $\mathbb{R}^n$. We say that f belongs to BMO, the space of functions of bounded mean oscillation, if

$$\sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|\mathrm{d}x<\infty. \tag{1}$$

Here the supremum is taken over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes, $|Q|$ denotes the Lebesgue measure of Q and

$$f_Q=\frac{1}{|Q|}\int_Q f(x)\mathrm{d}x$$

stands for the average of f over Q. A function f is said to have a bounded lower oscillation, if the average f_Q in (1) can be replaced by $\operatorname{essinf}_Q f$, the essential infimum of f over Q. That is, $f\in BLO$ if

$$\sup_Q \left[f_Q-\operatorname*{essinf}_Q f\right]<\infty. \tag{2}$$

The suprema in (1) and (2) are denoted by $\|f\|_{BMO}$ and $\|f\|_{BLO}$. One can consider the slightly different setting in which only the cubes Q within a given Q^0 are considered. In such a case, one often uses the notation $BMO(Q^0)$ and $BLO(Q^0)$, to indicate the corresponding base space. Another important modifications, the so-called dyadic BMO and BLO, correspond to the case when in (1) and (2) only the dyadic cubes (i.e., products of intervals of the form $(a2^{-m},(a+1)2^{-m}]$, where a, $m\in\mathbb{Z}$) are taken into consideration.

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If this is the case, we will add the superscript "d" and denote the modified spaces by BMO^d, BLO^d, etc.. Further probabilistic extensions will appear later in the text.

The BMO class was introduced by John and Nirenberg in [6] and it turned out to be one of the most important spaces in analysis and probability. One of its crucial features is that many classical operators (maximal, singular integral, etc.) map L^∞ into BMO. Another remarkable result, due to Fefferman [4], asserts that BMO is dual to the Hardy space H^1. We would also like to mention that BMO is important from the viewpoint of interpolation. For more on this interesting subject, see any textbook on harmonic analysis. The BLO class first appeared in the paper of Coifman and Rochberg [2], who used it to prove a decomposition property of BMO: any function from BMO can be written as a difference of two BLO functions. The BLO class arises naturally while studying the action of the Hardy-Littlewood maximal operator $\mathcal{M}$ on BMO spaces; for instance, Bennett [1] proved that $f \in BLO$ if and only if it is of the form $\mathcal{M}F + h$, where F is a function of bounded mean oscillation satisfying $\mathcal{M}F < \infty$ almost everywhere, and h is bounded. See also Korenovskii [7] for a variety of related results in this direction.

Let us recall a few simple properties of the class BLO. It is easy to see that it is a subset of BMO; more precisely, the bound $||f||_{BMO} \leq 2||f||_{BLO}$ holds true. Unlike BMO, the class BLO is not a linear space, as it is not even stable under multiplication by negative numbers ($-\log|x|$ is in BLO, but $\log|x|$ is not). Therefore, despite the notation, $||\cdot||_{BLO}$ is not a norm. However, it is easy to check that this functional is subadditive and positive-homogeneous. Furthermore, we have $BLO \cap (-BLO) = L^\infty$, which follows from the very definition. All the facts and properties formulated above have their counterparts in the dyadic case.

We will be interested in certain *weighted* inequalities for the spaces BLO in the dyadic context. Here and in what follows, the word 'weight' refers to a positive, locally integrable function w defined on a given base measure space (which can be $\mathbb{R}^n$, a fixed cube Q, or some probability space). Any weight gives rise to the corresponding measure on $\mathbb{R}^n$ (again denoted by w), which is defined by $w(A) = \int_A w\,\mathrm{d}x$. Note that the definitions of BMO and BLO make perfect sense if we replace the Lebesgue's measure by an arbitrary Borel measure; in particular, we may consider these spaces with respect to w.

There are two major problems which will be investigated in this paper:

· Characterize those weights w, for which the inclusion $BLO \hookrightarrow BLO(w)$ is bounded.

· Provide sharp lower and upper bounds for $\|\,\mathrm{Id}\,\|_{BLO \to BLO(w)}$ in terms of appropriate characteristics of w.

It turns out that the required characterization can be expressed in terms of the so-called A_∞ condition. A weight w is said to be the dyadic A_∞ weight (or belong to the dyadic

A_∞ class) if the so-called Wilson constant

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)\mathrm{d}x$$

is finite. Here M denotes the dyadic maximal operator, acting on locally integrable functions by

$$Mf = \sup_Q(|f|_Q \chi_Q),$$

and both suprema above are taken over all dyadic cubes in $\mathbb{R}^n$. We wlll prove the following two facts.

THEOREM 1.1. *Suppose that w is a weight on $\mathbb{R}^n$ such that $\|\operatorname{Id}\|_{BLO\to BLO(w)} < \infty$. Then w is an A_∞ weight and we have $[w]_{A_\infty} \leq \|\operatorname{Id}\|_{BLO\to BLO(w)}$.*

THEOREM 1.2. *For any dyadic A_∞ weight w on $\mathbb{R}^n$, we have $\|\operatorname{Id}\|_{BLO\to BLO(w)} \leq 2^d([w]_{A_\infty} - 1) + 1$. The estimate is sharp: for any $c \geq 1$, there is a dyadic A_∞ weight w on $\mathbb{R}^n$ for which $[w]_{A_\infty} = c$ and $\|\operatorname{Id}\|_{BLO\to BLO(w)} \geq 2^d([w]_{A_\infty} - 1) + 1$.*

The same theorems hold true if instead of $\mathbb{R}^n$, we restrict ourselves to functions and weights supported on a given base cube Q. Actually, we will study the above statements in the context of probability measures equipped with a tree-like structure. Here is the precise definition.

DEFINITION 1.3. Suppose that (X, μ) is a nonatomic probability space. A set $\mathcal{T}$ of measurable subsets of X will be called a tree, if the following conditions are satisfied:

(i) $X \in \mathcal{T}$ and for every $Q \in \mathcal{T}$ we have $\mu(Q) > 0$.

(ii) For every $Q \in \mathcal{T}$ there is a finite subset $C(Q) \subset \mathcal{T}$ containing at least two elements such that
   (a) the elements of $C(Q)$ are pairwise disjoint subsets of Q,
   (b) $Q = \bigcup C(Q)$.

(iii) $\mathcal{T} = \bigcup_{m\geq 0} \mathcal{T}^m$, where $\mathcal{T}^0 = \{X\}$ and $\mathcal{T}^{m+1} = \bigcup_{Q\in\mathcal{T}^m} C(Q)$.

(iv) We have $\lim_{m\to\infty} \sup_{Q\in\mathcal{T}^m} \mu(Q) = 0$.

DEFINITION 1.4. Let (X, μ) be a probability space with a tree $\mathcal{T}$ and let $\alpha \in (0, 1)$ be a fixed parameter. The tree $\mathcal{T}$ is called α-regular, if for any $Q \in \mathcal{T}$ and any child $Q' \in C(Q)$, we have $\mu(Q')/\mu(Q) \geq \alpha$.

For example, if we let X be a given dyadic cube in $\mathbb{R}^n$ with the normalized Lebesgue measure and the tree of its dyadic subcubes, then we obtain the localized setting discussed above. Furthermore, observe that the dyadic tree in this case is 2^{-n}-regular.

We define the probabilistic analogues of BLO classes and A_∞ weights using the same formulas as above: the role of dyadic cubes is played by the elements of the tree $\mathcal{T}$. So, a random variable f belongs to the space BLO, if

$$\|f\|_{BLO} = \sup_{Q\in\mathcal{T}} \Big[f_{Q,\mu} - \operatorname*{essinf}_Q f \Big] < \infty.$$

Here, of course, $f_{Q,\mu} = \frac{1}{\mu(Q)}\int_Q f\mathrm{d}\mu$ is the average of f over Q with respect to the probability measure μ, and the null-sets in the definition of the essential infimum are with respect to the measure μ. The A_∞ class is handled similarly, with the modification of Wilson constant given by

$$[w]_{A_\infty} = \sup_{Q\in\mathcal{T}} \frac{1}{w(Q)} \int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu,$$

where

$$M_{\mathcal{T}} f(x) = \sup_{Q\in\mathcal{T}} (f_{Q,\mu}\chi_Q)$$

is the $\mathcal{T}$-counterpart of the dyadic maximal function. We will usually skip the index μ and denote the average $f_{Q,\mu}$ just by f_Q, as in the Euclidean setting; this should not lead to any confusion, it will be clear from the context in which case we work.

We will establish the following version of Theorems 1.1 and 1.2.

THEOREM 1.5. *Let (X,μ) be a probability space with a tree $\mathcal{T}$. Suppose that w is a weight on X such that $\|\operatorname{Id}\|_{BLO\to BLO(w)} < \infty$. Then w is an A_∞ weight and we have $[w]_{A_\infty} \le \|\operatorname{Id}\|_{BLO\to BLO(w)}$. This estimate is sharp: for any $1 \le c < c'$, there is an A_∞ weight w on X such that $[w]_{A_\infty} = c$ and $\|\operatorname{Id}\|_{BLO\to BLO(w)} \le c'$.*

THEOREM 1.6. *Let (X,μ) be a probability space with an α-regular tree $\mathcal{T}$, where $\alpha \in (0,1)$ is a given parameter. Then for any dyadic A_∞ weight w on X, we have the estimate $\|\operatorname{Id}\|_{BLO\to BLO(w)} \le \alpha^{-1}([w]_{A_\infty} - 1) + 1$. The estimate is sharp: for any $c \ge 1$, there is a probability space (X,μ) with an α-regular tree $\mathcal{T}$ and an A_∞ weight w on X for which $[w]_{A_\infty} = c$ and $\|\operatorname{Id}\|_{BLO\to BLO(w)} \ge \alpha^{-1}([w]_{A_\infty} - 1) + 1$.*

A few words about the organization of the paper are in order. We will provide two proofs of Theorem 1.5: one of them will give a slightly worse estimate $[w]_{A_\infty} \le \|\operatorname{Id}\|_{BLO\to BLO(w)} + 1$, but it exploits a number of interesting facts and techniques, so we believe that it is worth including in the text. Theorems 1.1 and 1.5 are established in the next section. The final part contains the proof of Theorems 1.2 and 1.6.

2. PROOFS OF THEOREMS 1.1 AND 1.5

We start with the simple observation that it is enough to focus on Theorem 1.5; indeed, Theorem 1.1 is then a simple consequence. To see this, we pick an arbitrary dyadic cube Q, equip it with the tree of its dyadic subcubes and the normalized Lebegue measure. The application of Theorem 1.5 to this new probability space gives

$$[w]_{A^d_\infty(Q)} \le \|\operatorname{Id}\|_{BLO^d(Q)\to BLO^d(Q;w)} \le \|\operatorname{Id}\|_{BLO^d\to BLO^d(w)}$$

and it remains to take the supremum over Q.

2.1. **A special function with a concavity-type property.** Let $\alpha \in (0,1]$ be a fixed parameter. Consider the function $B : \mathbb{R}^4 \to \mathbb{R}$, given by $B(x,y,u,v) = (x-y-\alpha^{-1})v+yu$. This function enjoys the following concavity-type condition.

LEMMA 2.1. *Assume that x, y, u, v, d and e are real numbers satisfying $x-y \leq 1$, $u \leq v$ and one of the following conditions: (i) $d \leq \alpha^{-1}-1$ or (ii) $d > \alpha^{-1}-1$ and $e \leq 0$. Then we have the estimate*

$$\begin{aligned} &B(x+d, \max\{x+d-1,y\}, u+e, \max\{u+e,v\}) \\ &\qquad \leq B(x,y,u,v) + B_x(x,y,u,v)d + B_u(x,y,u,v)e. \end{aligned} \tag{3}$$

PROOF. The argument is straightforward: there are four possibilities to consider. If we have $x+d-1 \leq y$ and $u+e \leq v$, then the estimate is of the form

$$B(x+d,y,u+e,v) \leq B(x,y,u,v) + B_x(x,y,u,v)d + B_u(x,y,u,v)e$$

and actually, it becomes an equality: B is linear in x and u. If $x+d-1 > y$, $u+e \leq v$, then the inequality is equivalent to the trivial $(x+d-1-y)(u+e-v) \leq 0$. If $x+d-1 \leq y$ and $u+e > v$, then the claim reads $(x+d-y-\alpha^{-1})(u+e-v) \leq 0$, which follows from the bound $\alpha^{-1} \geq 1$. The final case is $x+d-1 > y$ and $u+e > v$, in which the assertion becomes $(x+d-y-\alpha^{-1})(u+e-v) \leq 0$. But then we necessarily have $e > 0$ (since $u \leq v$), so the assumption (i) must hold: $d \leq \alpha^{-1}-1$. Since $x-y \leq 1$, the estimate follows. ☐

2.2. **A lower bound for $\|\operatorname{Id}\|_{BLO \to BLO(w)}$, the weaker form.** We start with a lemma which follows from the results of Coifman and Rochberg [2]. We improve the statement by providing the best constant.

LEMMA 2.2. *Suppose that f is a nonnegative function. Then $\|\log M_{\mathcal{T}} f\|_{BLO} \leq 1$.*

PROOF. Recall the following classical maximal weak-type estimate for martingales: for any $Q \in \mathcal{T}$ and any integrable random variable φ supported on Q, we have

$$\lambda\mu(\{x \in Q : M_{\mathcal{T}}\varphi \geq \lambda\}) \leq \int_{\{x \in Q: M_{\mathcal{T}}\varphi \geq \lambda\}} \varphi \mathrm{d}\mu, \qquad \lambda > 0. \tag{4}$$

We proceed to the *BLO* bound for $\log M_{\mathcal{T}} f$. Fix $Q \in \mathcal{T}$. If $\int_Q f \mathrm{d}\mu = 0$, then there is nothing to prove: $M_{\mathcal{T}} f$ is constant on Q and the lower oscillation on Q is zero. So, suppose that $\int_Q f \mathrm{d}\mu > 0$ in the considerations below. We have

$$M_{\mathcal{T}} f = \max\left\{ \sup_{R: R \subseteq Q} \frac{1}{|R|}\int_R f \mathrm{d}\mu, \sup_{R: R \supseteq Q} \frac{1}{|R|}\int_R f \mathrm{d}\mu \right\} = \max\left\{ M_{\mathcal{T}}(f\chi_Q), y \right\}, \tag{5}$$

where $y = \sup_{R: R \supseteq Q} \frac{1}{|R|}\int_R f \mathrm{d}\mu$. Let us record the trivial bound

$$y \geq \frac{1}{|Q|}\int_Q f \mathrm{d}\mu > 0. \tag{6}$$

Using (4) for $\varphi = f\chi_Q$, we obtain

$$\begin{aligned}\int_Q \log M_{\mathcal{T}} f \mathrm{d}\mu &= \int_Q \log \max\{M_{\mathcal{T}}(f\chi_Q), y\}\mathrm{d}\mu \\ &= |Q|\log y + \int_Q \log \frac{\max\{M_{\mathcal{T}}(f\chi_Q), y\}}{y}\mathrm{d}\mu \\ &= |Q|\log y + \int_y^\infty \lambda^{-1}\mu(\{x \in Q : M_{\mathcal{T}}(f\chi_Q) \geq \lambda\})\mathrm{d}\lambda \\ &\leq |Q|\log y + \int_y^\infty \lambda^{-2}\int_{\{x\in Q: M_{\mathcal{T}}(f\chi_Q)\geq \lambda\}} f\chi_Q \mathrm{d}\mu\mathrm{d}\lambda \\ &= |Q|\log y + \int_Q f\left(\frac{1}{y} - \frac{1}{M_{\mathcal{T}}(f\chi_Q)}\right)\mathrm{d}\mu \\ &\leq |Q|\log y + \frac{\int_Q f\mathrm{d}\mu}{y} \leq |Q|\log y + |Q|,\end{aligned}$$

where in the last line we have used (6). Therefore, we will be done if we show that $y = \operatorname{essinf}_Q M_{\mathcal{T}} f$. To prove this, observe that by (4), we have $z\mu(\{x \in Q : M_{\mathcal{T}}(f\chi_Q) \geq z\}) \leq \int_Q f\mathrm{d}\mu$ for any $z > y$. Combining this with (6), we obtain

$$\mu(\{x \in Q : M_{\mathcal{T}}(f\chi_Q) \geq z\}) < |Q|$$

and hence $\operatorname{essinf}_Q M_{\mathcal{T}}(f\chi_Q) \leq z$. Letting $z \downarrow y$ we get $\operatorname{essinf}_Q M_{\mathcal{T}}(f\chi_Q) \leq y$, which together with (5) yields the desired identity $y = \operatorname{essinf}_Q M_{\mathcal{T}} f$. □

THEOREM 2.3. *Suppose that w is a weight on X such that $\|Id\|_{BLO\to BLO(w)} < \infty$. Then w belongs to the class A_∞ and $[w]_{A_\infty} \leq \|Id\|_{BLO\to BLO(w)} + 1$.*

PROOF. Fix $Q \in \mathcal{T}$. By (4) applied to the variable $w\chi_Q$, we get

$$\begin{aligned}\int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu &= |Q|w_Q + \int_{w_Q}^\infty \mu(\{x \in Q : M_{\mathcal{T}}(w\chi_Q) \geq \lambda\})\mathrm{d}\lambda \\ &\leq w(Q) + \int_{w_Q}^\infty \lambda^{-1}\int_{\{x\in Q: M_{\mathcal{T}}(w\chi_Q)\geq\lambda\}} w\mathrm{d}\mu\mathrm{d}\lambda \\ &= w(Q) + \int_Q w\log\frac{M_{\mathcal{T}}(w\chi_Q)}{w_Q}\mathrm{d}\mu\end{aligned}$$

and hence

$$\begin{aligned}\frac{1}{w(Q)}\int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu &\leq \frac{1}{w(Q)}\int_Q \log M_{\mathcal{T}}(w\chi_Q)w\mathrm{d}\mu - \log w_Q + 1 \\ &\leq \|\log M_{\mathcal{T}}(w\chi_Q)\|_{BLO(w)} + 1 \\ &\leq \|Id\|_{BLO\to BLO(w)}\|\log M_{\mathcal{T}}(w\chi_Q)\|_{BLO} + 1 \\ &\leq \|Id\|_{BLO\to BLO(w)} + 1,\end{aligned}$$

where the last passage follows from the previous lemma. This proves the claim, by taking the supremum over all $Q \in \mathcal{T}$. □

The above estimate can be improved, by removing the additive constant 1; we will show this in the next subsection.

2.3. **A lower bound for $\|\,\mathrm{Id}\,\|_{BLO\to BLO(w)}$, the stronger form.** We will establish the following fact, which is of independent interest. A random variable f is called $\mathcal{T}$-simple, if there is an integer N such that f is measurable with respect to the σ-algebra generated by $\mathcal{T}^N$.

THEOREM 2.4. *Suppose that w is a $\mathcal{T}$-simple weight. Then for any $Q \in \mathcal{T}$ there is a random variable f satisfying $\|f\|_{BLO} \leq 1$, $\operatorname{essinf}_Q f = 0$ and $\int_Q M_{\mathcal{T}}(w\chi_Q)d\mu = \int_Q fwd\mu$.*

PROOF. By the definition of the maximal function and the simplicity of w, for each $\omega \in Q$ there is $R = R(\omega) \in \mathcal{T}^0 \cup \mathcal{T}^1 \cup \mathcal{T}^2 \cup \cdots \cup \mathcal{T}^N$ which satisfies $\omega \in R \subseteq Q$ and $M_{\mathcal{T}}(w\chi_Q)(\omega) = w_R$ (simply speaking: the maximal function is equal to the average over some set R). In general, there may be many sets $R(\omega)$ with this property - if this is the case, we pick the set having the largest probability. Now, for any $S \in \mathcal{T}$, $S \subseteq Q$, define $E(S) = \{\omega \in Q : R(\omega) = S\}$. Then $\{E(R)\}_{R\in\mathcal{T},\,R\subseteq Q}$ is a collection of pairwise disjoint sets satisfying $E(R) \subseteq R$ and $\bigcup E(R) = Q$; furthermore, $E(R) = \emptyset$ if $R \in \mathcal{T}^n$ for some $n > N$. Therefore, we may write

$$\int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu = \sum_{\substack{R\in\mathcal{T},\\ R\subseteq Q}} w_R \cdot \mu(E(R)) = \int_Q \sum_{\substack{R\subseteq Q,\\ R\in\mathcal{T}}} \frac{\mu(E(R))}{\mu(R)}\chi_R \cdot w\mathrm{d}\mu.$$

Introduce the nonnegative $\mathcal{T}$-simple random variable $f = \sum_{R\subseteq Q, R\in\mathcal{T}} \frac{\mu(E(R))}{\mu(R)}\chi_R$; then we have $\int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu = \int_Q fw\mathrm{d}\mu$ by the above calculation. Furthermore, this function satisfies $\|f\|_{BLO} \leq 1$. To check this, fix an arbitrary element S of $\mathcal{T}$. If $Q \cap S = \emptyset$ or $Q \subseteq S$, then

$$f_S \leq f_Q = \frac{1}{\mu(Q)} \sum_{\substack{R\subseteq Q,\\ R\in\mathcal{T}}} \mu(E(R)) = 1$$

and $\operatorname{essinf}_S f \geq 0$, so the BLO condition holds for such S. On the other hand, if $S \subset Q$, then

$$\begin{aligned} f_S &= \frac{1}{\mu(S)}\left[\sum_{\substack{R\subsetneq S,\\ R\in\mathcal{T}}} \mu(E(R)) + \sum_{\substack{S\subseteq R\subseteq Q,\\ R\in\mathcal{T}}} \frac{\mu(E(R))}{\mu(R)}\mu(S)\right] \\ &\leq \frac{1}{\mu(S)}\left[\mu(S) + \mu(S)\cdot \sum_{\substack{S\subset R\subseteq Q,\\ R\in\mathcal{T}}} \frac{\mu(E(R))}{\mu(R)}\right] \leq 1 + \operatorname*{essinf}_S f. \end{aligned}$$

Finally, we modify f slightly to ensure the condition $\operatorname{essinf}_Q f = 0$. We construct inductively the sequence $I_0 \supset I_1 \supset I_2 \supset \cdots$ such that $I_0 = Q$ and for each n, I_{n+1} is a child of I_n satisfying $f_{I_n} \geq f_{I_{n+1}}$. Since f and w are $\mathcal{T}$-simple, there is an integer k such that f and w

are constant on I_k (in particular, the sequence (f_{I_n}) stabilizes: $f_{I_k} = f_{I_{k+1}} = f_{I_{k+2}} = \cdots$). We modify f on I_k: let

$$\tilde{f} = f_{I_k} \cdot \frac{\mu(I_k)\chi_{I_{k+1}}}{\mu(I_{k+1})} + f\chi_{X\setminus I_k}.$$

This function satisfies $\operatorname{essinf}_Q \tilde{f} = 0$: it is nonnegative and vanishes on $I_k \setminus I_{k+1}$. Since w is constant on I_k, we have $\int_{I_k} \tilde{f} w \mathrm{d}\mu = \int_{I_k} f w \mathrm{d}\mu$; furthermore, $f = \tilde{f}$ outside I_k, so

$$\int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu = \int_Q f w \mathrm{d}\mu = \int_Q \tilde{f} w \mathrm{d}\mu.$$

Finally, we have $\|\tilde{f}\|_{BLO} \leq 1$. To check this, fix an arbitrary element S of $\mathcal{T}$. If S does not intersect I_k, then $\tilde{f}_S = f_S$ and $\operatorname{essinf}_S \tilde{f} = \operatorname{essinf}_S f$, and hence $\tilde{f}_S - \operatorname{essinf}_S \tilde{f} = f_S - \operatorname{essinf}_S f \leq 1$ as we have shown above. On the other hand, if $S \cap I_k \neq \emptyset$, then we have two options. If S is strictly contained in I_k, then f is constant on S and hence $\tilde{f}_S - \operatorname{essinf}_S \tilde{f} = 0$; if, finally, $I_k \subseteq S$, then $S = I_m$ for some $0 \leq m \leq k$ and hence

$$\tilde{f}_S - \operatorname*{essinf}_S \tilde{f} = \tilde{f}_{I_m} = f_{I_m} \leq f_{I_0} = f_Q \leq 1,$$

as we computed above. This proves the desired assertion. □

As an immediate corollary, we get the sharp relation between A_∞ and BLO.

THEOREM 2.5. *For any weight w, we have*

$$[w]_{A_\infty} \leq \|Id\|_{BLO\to BLO(w)}. \tag{7}$$

The above inequality is sharp: for any $1 \leq c \leq c'$ there is a probability space with a tree which supports a weight w satisfying $[w]_{A_\infty} = c$ and $\|Id\|_{BLO\to BLO(w)} \leq c'$.

Proof of (7). Fix $Q \in \mathcal{T}$ and an integer N. Let us apply the previous theorem to the simple weight $w^N := \mathcal{E}_N w$, where $\mathcal{E}_N$ denotes the conditional expectation with respect to $\mathcal{T}^N$. As the result, we obtain a random variable f satisfying $\|f\|_{BLO} \leq 1$, $\operatorname{essinf}_Q f = 0$ and

$$\int_Q M_{\mathcal{T}}(w^N\chi_Q)\mathrm{d}\mu = \int_Q f w^N \mathrm{d}\mu - w(Q) \operatorname*{essinf}_Q f.$$

Furthermore, picking an appropriate child I_{k+1} in the previous proof, we may ensure that $\int_Q f w^N \mathrm{d}\mu - w(Q) \operatorname{essinf}_Q f \leq \int_Q f w \mathrm{d}\mu - w(Q) \operatorname{essinf}_Q f$. Therefore, we get

$$\frac{1}{w(Q)} \int_Q M_{\mathcal{T}}(w^N\chi_Q)\mathrm{d}\mu \leq \|f\|_{BLO(w)} \leq \|Id\|_{BLO\to BLO(w)} \|f\|_{BLO} \leq \|Id\|_{BLO\to BLO(w)}.$$

It remains to note that the sequence $(M_{\mathcal{T}}(w^N\chi_Q))_{N\geq 1}$ increases almost surely to $M_{\mathcal{T}}(w\chi_Q)$. Thus, by Lebesgue's monotone convergence theorem, we obtain

$$\frac{1}{w(Q)} \int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu \leq \|Id\|_{BLO\to BLO(w)}$$

and taking the supremum with respect to Q completes the proof. □

2.4. **Sharpness of** (7)**.** If $c = 1$, then any constant weight w gives equality in (7). So, from now on, we assume that c is bigger than 1. Fix a parameter $\alpha \in (0, 1)$ (which will eventually be sent to zero) and put $\gamma = \alpha^{-1} + c^{-1} - (\alpha c)^{-1} > 1$.

We split the reasoning into a few steps.

Step 1. Definitions.

Consider the probability space $([0, 1), \mathcal{B}([0, 1)), |\cdot|)$ with the tree given by the following inductive procedure. We set $\mathcal{T}^0 = \{[0, 1)\}$ and, given an interval $I \in \mathcal{T}$, we split it into two intervals: the left part I_- and the right part I_+, with $|I_-| = \alpha|I|$ and $|I_+| = (1-\alpha)|I|$. For the sake of completeness, let us assume that both I_- and I_+ are closed from the left and open from the right. The collection of all such $I_\pm$ (obtained from different $I \in \mathcal{T}^n$) forms $\mathcal{T}^{n+1}$. Note that the filtration is $\min\{\alpha, 1-\alpha\}$-regular, directly from the definition. Obviously, for each $n \geq 0$, the interval $I_n = [0, \alpha^n)$ belongs to $\mathcal{T}^n$. Finally, introduce $w : [0, 1) \to [0, \infty)$ by the formula

$$w = \sum_{n=0}^{\infty} \gamma^n \chi_{I_n \setminus I_{n+1}}.$$

Step 2. The equality $[w]_{A_\infty} = c$*.* Since $\gamma > 1$, the weight w is a decreasing function on $[0, 1)$. Consequently, for $\omega \in (0, 1)$, when computing $M_\mathcal{T} w(\omega)$, one has to take the average over the smallest I_n which contains ω. That is, if we put $m = \max\{n : \alpha^n > \omega\}$, then we have

$$M_\mathcal{T} w(\omega) = \frac{1}{|I_m|} \int_{I_m} w \mathrm{d}\mu = \alpha^{-m} \sum_{n=m}^{\infty} \gamma^n (\alpha^n - \alpha^{n+1}) = \gamma^m \cdot \frac{1-\alpha}{1-\alpha\gamma} = c\gamma^m. \tag{8}$$

In other words, we have $M_\mathcal{T} w = cw$ on $(0, 1)$ and hence $[w]_{A_\infty} = c$. Indeed, on one hand we have $[w]_{A_\infty} \geq \frac{1}{w([0,1))} \int_{[0,1)} M_\mathcal{T} w \mathrm{d}\mu = c$, and on the other hand, for any $Q \in \mathcal{T}$,

$$\frac{1}{w(Q)} \int_Q M_\mathcal{T}(w\chi_Q) \mathrm{d}\mu \leq \frac{1}{w(Q)} \int_Q M_\mathcal{T} w \mathrm{d}\mu = c.$$

Step 3. Further notation. Let f be an integrable random variable satisfying $\|f\|_{BLO} = 1$. Let Q be an arbitrary interval from $\mathcal{T}$; then there exists m such that $Q \in \mathcal{T}^m$. Define the sequences $(x_n)_{n \geq m}$, $(y_n)_{n \geq m}$, $(u_n)_{n \geq m}$ and $(v_n)_{n \geq m}$ of random variables as follows: for each n set

$$x_n(\omega) = f_{Q_n(\omega)}, \quad y_n(\omega) = \operatorname*{essinf}_{Q_n(\omega)} f, \quad u_n(\omega) = w_{Q_n(\omega)}, \quad v_n(\omega) = \max_{m \leq k \leq n} u_k(\omega),$$

where $Q_n(\omega)$ is the unique element of $\mathcal{T}^n$ which contains ω. There is a nice probabilitic interpretation of these sequences. Namely, $(x_n)_{n \geq m}$ and $(u_n)_{n \geq m}$ are the martingales generated by f and w with respect to the filtration $(\sigma(\mathcal{T}^n))_{n \geq m}$. Furthermore, $(v_n)_{n \geq m}$ is the maximal function of $(u_n)_{n \geq m}$, and $(y_n)_{n \geq m}$ can be regarded as a variant of the minimal function of $(x_n)_{n \geq m}$.

We conclude this part by recording some important observations about the behavior of the sequences $(x_n)_{n\geq m}$, $(y_n)_{n\geq m}$ and $(w_n)_{n\geq m}$. Fix $n\geq m$. First, since $\|f\|_{BLO}\leq 1$, we have $x_n\leq y_n+1$ almost surely. To present the second property, fix an interval $I\in\mathcal{T}^n$ and denote its left and right children in $\mathcal{T}^{n+1}$ by I_- and I_+. Then, using $\|f\|_{BLO}\leq 1$ again, we get

$$f_I\geq \frac{|I_-|}{|I|}f_{I_-}+\frac{|I_+|}{|I|}\operatorname*{essinf}_I f\geq \frac{|I_-|}{|I|}f_{I_-}+\frac{|I_+|}{|I|}(f_I-1),$$

or equivalently, $f_{I_-}-f_I\leq |I|/|I_-|-1=\alpha^{-1}-1$. Furthermore, by the aforementioned monotonicity of w, we immediately obtain $w_{I_+}-w_I\leq 0$. In other words, we have $x_{n+1}-x_n\leq\alpha^{-1}-1$ or $u_{n+1}-u_n\leq 0$ almost surely.

Step 4. Implementing the Bellman function. Let B be the special function of §2.1, with the parameter $K=\alpha^{-1}$. We use (3) with $x=x_n$, $y=y_n$, $u=u_n$, $v=v_n$ and $d=x_{n+1}-x_n$, $e=u_{n+1}-u_n$. Note that we have $d\leq K-1$; or $d>K-1$ and $e\leq 0$, so the assumptions are satisfied and the application is permitted. As the result, we obtain

$$\begin{aligned}&B\big(x_{n+1},\max\{x_{n+1}-1,y_n\},u_{n+1},\max\{u_{n+1},v_n\}\big)\\&\qquad\leq B(x_n,y_n,u_n,v_n)+B_x(x_n,y_n,u_n,v_n)d_{n+1}+B_u(x_n,y_n,u_n,v_n)e_{n+1}.\end{aligned}$$

However, we have $\max\{x_{n+1}-1,y_n\}\leq y_{n+1}$ and $\max\{u_{n+1},v_n\}=v_{n+1}$, so

$$B(x_{n+1},\max\{x_{n+1}-1,y_n\},u_{n+1},\max\{u_{n+1},v_n\}\big)\geq B(x_{n+1},y_{n+1},u_{n+1},v_{n+1})$$

(indeed: we have $B_y(x_{n+1},y,u_{n+1},v_{n+1})=u_{n+1}-v_{n+1}\leq 0$), which combined with the previous estimate yields

$$\begin{aligned}&B\big(x_{n+1},y_{n+1},u_{n+1},v_{n+1}\big)\\&\qquad\leq B(x_n,y_n,u_n,v_n)+B_x(x_n,y_n,u_n,v_n)d_{n+1}+B_u(x_n,y_n,u_n,v_n)e_{n+1}.\end{aligned}$$

Now, pick an arbitrary $R\in\mathcal{T}^n$ and integrate both sides over R (with respect to the measure μ). The terms $B(x_n,y_n,u_n,v_n)$, $B_x(x_n,y_n,u_n,v_n)$ and $B_u(x_n,y_n,u_n,v_n)$ are constant over R; furthermore, we have $\int_R d\mathrm{d}\mu=\int_R x_{n+1}\mathrm{d}\mu-\int_Q x_n\mathrm{d}\mu=0$ and similarly $\int_R e\mathrm{d}\mu=0$. Consequently, we obtain

$$\int_R B\big(x_{n+1},y_{n+1},u_{n+1},v_{n+1}\big)\mathrm{d}\mu\leq\int_R B(x_n,y_n,u_n,v_n)\mathrm{d}\mu.$$

Summing over all $R\in\mathcal{T}^n$ which are contained in Q (which was fixed at the beginning of the proof) we finally get

$$\int_Q B\big(x_{n+1},y_{n+1},u_{n+1},v_{n+1}\big)\mathrm{d}\mu\leq\int_Q B(x_n,y_n,u_n,v_n)\mathrm{d}\mu,$$

which in particular implies that for each $n\geq m$ we have

$$\int_Q B\big(x_n,y_n,u_n,v_n\big)\mathrm{d}\mu\leq\int_Q B(x_m,y_m,u_m,v_m)\mathrm{d}\mu=|Q|B(x_m,y_m,u_m,v_m).\tag{9}$$

Step 5. Limiting arguments and completion of the proof. To analyze the right-hand side of (9), note that $u_m = v_m$ and $x_m - y_m \leq 1$, so

$$B(x_m, y_m, u_m, v_m) = (x_m - y_m - \alpha^{-1})u_m + y_m u_m \leq (\operatorname*{essinf}_Q f + 1 - \alpha^{-1})w_Q.$$

Next, by martingale limit theorems, we have $x_n \to f$, $y_n \uparrow f$, $u_n \to w$ and $v_n \uparrow M_{\mathcal{T}}(w\chi_Q)$ almost surely. Consequently, by Fatou's lemma,

$$\liminf_{n\to\infty} \int_Q (x_n - y_n)v_n \mathrm{d}\mu \geq 0, \qquad \liminf_{n\to\infty} \int_Q y_n u_n \mathrm{d}\mu \geq \int_Q fw\mathrm{d}\mu$$

(in the second limit we have used the fact that y_n is bounded from below by y_m) and hence, by Lebesgue's monotone convergence theorem,

$$\lim_{n\to\infty} \int_Q v_n \mathrm{d}\mu = \int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu.$$

Putting all these facts together and plugging into (9), we arrive at

$$\int_Q fw\mathrm{d}\mu - \alpha^{-1}\int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu \leq |Q| \cdot (\operatorname*{essinf}_Q f + 1 - \alpha^{-1})w_Q,$$

which is equivalent to

$$\frac{1}{w(Q)}\int_Q fw\mathrm{d}\mu - \operatorname*{essinf}_Q f \leq \alpha^{-1}\left(\frac{1}{w(Q)}\int_Q M_{\mathcal{T}}(w\chi_Q)\mathrm{d}\mu - 1\right) + 1 \leq \alpha^{-1}[w]_{A_\infty}.$$

Since Q was arbitrary and $\|f\|_{BLO} = 1$, we obtain $\|f\|_{BLO(w)} \leq \alpha^{-1}[w]_{A_\infty}\|f\|_{BLO}$ and hence $\|\operatorname{Id}\|_{BLO\to BLO(w)} \leq \alpha^{-1}[w]_{A_\infty}$. It remains to let $\alpha \uparrow 1$ to get the claim.

3. Proofs of Theorems 1.2 and 1.6

A similar reasoning to that above yields the following statement.

THEOREM 3.1. *Suppose that (X, μ) is a probability space equipped with an α-regular filtration. Then we have the sharp estimate*

$$\|\operatorname{Id}\|_{BLO\to BLO(w)} \leq \alpha^{-1}([w]_{A_\infty} - 1) + 1. \tag{10}$$

PROOF. Let f be an integrable random variable satisfying $\|f\|_{BLO} = 1$ and let w be an arbitrary weight belonging to the class A_∞. Fix an element $Q \in \mathcal{T}$ and let m denote the generation of the tree to which it belongs: $Q \in \mathcal{T}^m$. Define the sequences $(x_n)_{n\geq m}$, $(y_n)_{n\geq m}$, $(u_n)_{n\geq m}$ and $(v_n)_{n\geq m}$ as above.

Let us study the evolution properties of these sequences. As before, since $\|f\|_{BLO} \leq 1$, we have $x_n \leq y_n + 1$ almost surely. Furthermore, fix $n \geq m$, an element $R \in \mathcal{T}^n$ and denote its children in $\mathcal{T}^{n+1}$ by R_1, R_2, ..., R_k. Again by $\|f\|_{BLO} \leq 1$, we see that for each $1 \leq j \leq k$,

$$f_R \geq \frac{|R_j|}{|R|}f_{R_j} + \left(1 - \frac{|R_j|}{|R|}\right)\operatorname*{essinf}_R f \geq \frac{|R_j|}{|R|}f_{R_j} + \left(1 - \frac{|R_j|}{|R|}\right)(f_R - 1),$$

which can be rewritten in the form $f_{R_j} - f_R \leq |R|/|R_j| - 1 \leq \alpha^{-1} - 1$. Hence $x_{n+1} - x_n \leq \alpha^{-1} - 1$ almost surely, since R was chosen arbitrarily.

The remaining part of the proof goes along the same lines and exploits the Bellman function B corresponding to $K = \alpha^{-1}$. We use the concavity condition (3) with the assumption (i) only (which is guaranteed by the almost sure estimate $x_{n+1} - x_n \leq \alpha^{-1} - 1$ established above). We omit the straightforward repetition. As the result, we get

$$\frac{1}{w(Q)} \int_Q fw \mathrm{d}\mu - \operatorname*{essinf}_Q f \leq \alpha^{-1} \left(\frac{1}{w(Q)} \int_Q M_{\mathcal{T}}(w\chi_Q) \mathrm{d}\mu - 1 \right) + 1.$$

This obviously gives the claim. □

Now we will prove that the constant $\alpha^{-1}([w]_{A_\infty} - 1) + 1$ cannot be improved.

Sharpness of (10). If $c = 1$, then there is nothing to prove: the weight w is constant. So, let us assume that $c > 1$, fix $\alpha \in (0, 1/2]$ and consider the parameter $\gamma = \alpha^{-1} + c^{-1} - (\alpha c)^{-1}$. We consider the probability space as in the proof of the sharpness of (7). Note that the filtration studied there is α-regular, directly from the definition. We also distinguish the interval $I_n = [0, \alpha^n)$ belonging to $\mathcal{T}^n$, $n = 0, 1, 2, \ldots$.

Let w, $f : [0, 1) \to [0, \infty)$ be given by

$$w = \sum_{n=0}^{\infty} \gamma^n \chi_{I_n \setminus I_{n+1}}, \qquad f = \sum_{n=0}^{\infty} n \chi_{I_n \setminus I_{n+1}}.$$

We have already checked above that $[w]_{A_\infty} = c$. Furthermore, as we show now, we have $\|f\|_{BLO} = \alpha/(1 - \alpha)$. To this end, fix an arbitrary element $Q \in \mathcal{T}$. If $0 \notin Q$, then there exists an integer m such that $Q \subseteq I_m \setminus I_{m+1}$. Therefore, f is constant on Q and hence $f_Q - \operatorname{liminf}_Q f = 0$. On the other hand, if $0 \in Q$, then $Q = I_m$ for some m, so $\operatorname{essinf}_Q f = m$ and

$$f_Q = \frac{1}{|Q|} \sum_{n=m} n |I_n \setminus I_{n+1}| = \alpha^{-m} \sum_{n=m}^{\infty} n\alpha^n (1 - \alpha) = m + \frac{\alpha}{1 - \alpha}.$$

Hence $f_Q - \operatorname{liminf}_Q f = \alpha/(1 - \alpha)$, which yields the identity $\|f\|_{BLO} = \alpha/(1 - \alpha)$.

It remains to compute that

$$\begin{aligned}
\|f\|_{BLO(w)} &\geq \frac{1}{w([0, 1))} \int_{[0,1)} fw \mathrm{d}\mu - \operatorname*{essinf}_{[0,1)} f \\
&= \frac{1}{c} \sum_{n=0}^{\infty} n\gamma^n (\alpha^n - \alpha^{n+1}) = \frac{\alpha(1 - \alpha)\gamma}{c(1 - \alpha\gamma)^2} \\
&= \Big(\alpha^{-1}(c - 1) + 1\Big) \cdot \frac{\alpha}{1 - \alpha} = \Big(\alpha^{-1}([w]_{A_\infty} - 1) + 1\Big) \|f\|_{BLO}.
\end{aligned}$$

This yields the desired sharpness. □

It remains to prove the final of the statements formulated in the introductory section.

Proof of Theorem 1.2. Fix an arbitrary dyadic cube Q and equip it with the tree of its dyadic subcubes and the normalized Lebesgue measure. Pick arbitrary weight $w \in A_\infty^d$ and $f \in BLO^d$. Then the restrictions $w|_Q$ and $f|_Q$ belong to $A_\infty^d(Q)$ and $BLO^d(Q)$; we actually have $[w]_{A_\infty^d(Q)} \leq [w]_{A_\infty^d}$ and $\|f\|_{BLO^d(Q)} \leq \|f\|_{BLO^d}$, directly from the A_∞ and BLO conditions. Therefore, by Theorem 1.6,

$$\|f\|_{BLO^d(Q;w)} \leq \Big(2^d([w]_{A_\infty^d(Q)} - 1) + 1\Big)\|f\|_{BLO^d(Q)} \leq \Big(2^d([w]_{A_\infty^d} - 1) + 1\Big)\|f\|_{BLO^d}.$$

Letting $|Q| \to \infty$ we get the desired estimate. To see that this estimate is sharp, we repeat the above construction, with the probability space $([0,1)^N, \mathcal{B}([0,1)^N), |\cdot|)$ and its dyadic filtration, and the intervals I_n replaced by the cubes $[0, 2^{-n})^N$. Then we extend w and f to the whole $\mathbb{R}^N$ by the corresponding averages on $[0,1)^N$: we set $w|_{\mathbb{R}^n \setminus [0,1)^N} = w_{[0,1)^N}$ and $f|_{\mathbb{R}^n \setminus [0,1)^N} = f_{[0,1)^N}$. This guarantees that the A_∞^d and BLO^d constants will remain unchanged. Hence the estimate

$$\|f\|_{BLO^d(w)} \geq \Big(\alpha^{-1}([w]_{A_\infty} - 1) + 1\Big)\|f\|_{BLO}$$

holds true. This completes the proof. □

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