MAXIMAL INEQUALITIES FOR FUNCTIONS OF BOUNDED LOWER OSCILLATION

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Abstract. We introduce a method which can be used to establish sharp maximal estimates for functions of bounded lower oscillation. The technique allows to deduce such estimates from the existence of certain special functions, and can be regarded as a version of Bellman function method, which has gained considerable interest in the recent literature. As an application, we establish a sharp exponential bound, which can be regarded as a version of integral John-Nirenberg inequality for BLO functions.

1. Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued locally integrable function. We say that f belongs to BMO, the space of functions of bounded mean oscillation, if

$$\sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|\mathrm{d}x < \infty, \tag{1.1}$$

where the supremum is over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes, $|Q|$ denotes the volume of Q, and

$$f_Q = \frac{1}{|Q|}\int_Q f(x)\mathrm{d}x$$

is the mean of f over Q. A function f is said to have bounded lower oscillation, if the term f_Q in (1.1) can be replaced by $\operatorname{ess\,inf}_Q f$, the esstential infimum of f over Q. That is, $f \in BLO$ if

$$\sup_Q \left[f_Q - \operatorname*{ess\,inf}_Q f \right] < \infty. \tag{1.2}$$

The suprema in (1.1) and (1.2) are denoted by $||f||_{BMO}$ and $||f||_{BLO}$. We will consider a slightly less restrictive setting in which only the cubes Q within a given Q^0 are considered; to stress the dependence on Q^0, we will use the notation $BMO(Q^0)$ and $BLO(Q^0)$.

Sometimes it is more convenient to work with dyadic versions of BMO and BLO. This corresponds to the setting when in (1.1) and (1.2) only the dyadic cubes (i.e., products of intervals of the form $(a2^{-n}, (a+1)2^{-n}]$, where a, $n \in \mathbb{Z}$) are taken into consideration. If this is the case, we will add the superscript "d" and write BMO^d, BLO^d, etc., to stress that we deal with this wider class of functions.

The BMO class was introduced by John and Nirenberg in [7] and since then, it has played an important role in analysis and probability. One of its important

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features is that many classical operators (maximal, singular integral, etc.) map L^∞ into BMO. Another remarkable result, due to Fefferman [4], identifies BMO as a dual to the Hardy space H^1. The functions of bounded mean oscillation have very strong integrability properties (see e.g. [7]). In particular, we have the following celebrated result, the so-called integral John-Nirenberg inequality: if $f \in BMO(\mathbb{R}^n)$, then there is a constant c depending on the dimension n and the norm $||f||_{BMO(\mathbb{R}^n)}$ such that

$$\sup_Q \left\{ \int_Q \exp\left(c\,|f(x) - f_Q|\right) \mathrm{d}x \right\} < \infty. \tag{1.3}$$

The BLO class first appeared in the paper of Coifman and Rochberg [2], who used it to prove a decomposition property of BMO: any function from BMO can be written as a difference of two BLO functions. This decomposition has an interesting counterpart in the theory of Muckenhoupt weights: since $BMO = \{\alpha \log \omega : \alpha \geq 0, \omega \in A_2\}$ and $BLO = \{\alpha \log \omega : \alpha \geq 0, \omega \in A_1\}$ (see [2]), the statement $BMO = BLO - BLO$ can be regarded as the logarithm of the factorization $A_2 = A_1/A_1$ of Jones [8]; see also [5] and [6]. The BLO class arises naturally while studying the action of the Hardy-Littlewood maximal operator $\mathcal{M}$ on BMO spaces; for instance, Bennett [1] proved that $f \in BLO$ if and only if it is of the form $\mathcal{M}F + h$, where F is a function of bounded mean oscillation satisfying $\mathcal{M}F < \infty$ almost everywhere, and h is bounded. See also Korenovskii [9] for a variety of related results in this direction.

Let us recall a few simple properties of the class BLO. It is easy to see that it is a subspace of BMO; more precisely, the bound $||f||_{BMO} \leq 2||f||_{BLO}$ holds true. Unlike BMO, the class BLO is not a linear space, as it is not even stable under multiplication by negative numbers ($-\log|x|$ is in BLO, but $\log|x|$ is not). Therefore, despite the notation, $||\cdot||_{BLO}$ is not a norm. However, it is easy to check that this functional is subadditive and positive-homogeneous. Furthermore, we have $BLO \cap (-BLO) = L^\infty$, which follows from the very definition. We would like to stress that all the facts and properties formulated above have their counterparts in the dyadic case.

Our motivation comes from the question about a maximal version of (1.3) for the class BLO^d. We will introduce a general method which can be used to prove estimates of this type. The technique will allow us to deduce such inequalities from the existence of a certain special function, which enjoys appropriate majorization and concavity properties. This is the so-called Bellman function method, and its numerous variants and modifications have been studied intensively in the recent literature; we refer the interested reader to the papers [3], [10]−[16]. Consult also the references therein.

To state our main result, we need to introduce the notion of dyadic maximal operator. Let $Q \subset \mathbb{R}^n$ be a fixed dyadic cube and let $f : Q \to \mathbb{R}$ be a measurable function. We define the operator $\mathcal{M}_Q^d$ by the formula

$$\mathcal{M}_Q^d f(x) = \sup_{Q'} \left\{ \frac{1}{|Q'|} \int_{Q'} f(x) \mathrm{d}x \right\},$$

where the supremum is taken over all dyadic cubes Q' contained within Q. We will establish the following statement.

Theorem 1.1. *Let $n \geq 1$ be a given integer and let $Q \subset \mathbb{R}^n$ be a fixed dyadic cube. Then for any function $f \in BLO^d(Q)$ and any number β satisfying*

$$0 < \beta < \frac{n \log 2}{(2^n - 1)||f||_{BLO^d(Q)}},$$

we have

$$\frac{1}{|Q|}\int_Q \exp\left\{\beta\left(\mathcal{M}_Q^d f(x) - f_Q\right)\right\} dx \leq \frac{2^n - 1}{2^n - \exp\left(\beta(2^n - 1)||f||_{BLO^d(Q)}\right)}. \tag{1.4}$$

The constant on the right is the best possible for each n and each β. If $\beta \geq n \log 2/(2^n - 1)$, then there is a function f with $||f||_{BLO([0,1]^n)} \leq 1$ for which the left-hand side of (1.4) *is infinite.*

Actually, we will study the above result in a slightly more general setting. Instead of working in $\mathbb{R}^n$ equipped with the lattice of dyadic cubes, we will study analogous theorem for general measure spaces equipped with a tree-like structure. This framework is described in detail in the next section. Then, in Section 3, we present the appropriate modification of Bellman function method which allows the study of maximal inequalities for BLO^d. The final part of the paper contains the proof of our main results, Theorem 1.1 and Theorem 2.5, which is stated below.

2. Measure spaces with a tree-like structure

In this section we introduce the basic setup for our further considerations. Assume that $(X, \mathcal{F}, \mu)$ is a given non-atomic probability space, and let us equip it with an additional tree structure.

Definition 2.1. Let $\alpha \in (0, 1]$ be a fixed number. A sequence $\mathcal{T} = (\mathcal{T}_n)_{n\geq 0}$ of partitions of X is said to be α-splitting, if the following conditions hold.

(i) We have $\mathcal{T}_0 = \{X\}$ and $\mathcal{T}_n \subset \mathcal{F}$ for all n.
(ii) For any $n \geq 0$ and any $E \in \mathcal{T}_n$ there are E_1, E_2, ..., $E_m \in \mathcal{T}_{n+1}$ whose union is E and such that $\mu(E_i)/\mu(E) \geq \alpha$ for all i.

We note here that in general, the number m in (ii) may be different for different E.

Example. Assume that $X = (0, 1]^n$ is the unit cube of $\mathbb{R}^n$ with Borel subsets and Lebesgue's measure. Let $\mathcal{T}_k$ be a collection of all dyadic cubes of volume 2^{-kn}, contained in X. Then $\mathcal{T} = (\mathcal{T}_n)_{n\geq 0}$ is 2^{-n}-splitting.

In what follows, we may restrict ourselves to $\alpha \leq 1/2$, since for $\alpha > 1/2$ there is only one, trivial α-splitting tree $\mathcal{T} = (\{X\}, \{X\}, \{X\}, \ldots)$, for which all inequalities are evident. Any α-splitting tree gives rise to the appropriate maximal operator and the appropriate BLO class. Let us introduce these two objects.

Definition 2.2. Given a probability space $(X, \mathcal{F}, \mu)$ with a sequence $\mathcal{T}$ as above, we define the corresponding (one-sided) maximal operator $\mathcal{M}_\mathcal{T}$ as follows. For a given $f \in L^1(X, \mathcal{F}, \mu)$, let

$$\mathcal{M}_\mathcal{T} f(x) = \sup\left\{\frac{1}{\mu(E)}\int_E f \mathrm{d}\mu\right\},$$

where the supremum is taken over all E which contain x and belong to $\mathcal{T}_n$ for some $n \geq 0$. We will also use the notation $\mathcal{M}_\mathcal{T}^n$ for the truncated maximal operator, associated with $\mathcal{T}^n = (\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{n-1}, \mathcal{T}_n, \mathcal{T}_n, \mathcal{T}_n, \ldots)$.

Definition 2.3. A μ-locally integrable function $f : X \to \mathbb{R}$ belongs to $BLO(\mathcal{T})$, the BLO class induced by $\mathcal{T}$, if

$$||f||_{BLO(\mathcal{T})} = \sup\left\{ \frac{1}{\mu(E)}\int_E f(x)\mathrm{d}\mu(x) - \operatorname*{essinf}_{x\in E} f(x)\ :\ E \in \bigcup_{n\geq 0}\mathcal{T}_n \right\} < \infty. \tag{2.1}$$

The condition (2.1) can be expressed without the use of the essential infimum, by means of the maximal operator.

Lemma 2.4. *A locally integrable function $f : X \to \mathbb{R}$ belongs to $BLO(\mathcal{T})$ if and only if there is a finite constant C such that for μ-almost all $y \in X$ we have*

$$\mathcal{M}_{\mathcal{T}} f(y) \leq f(y) + C. \tag{2.2}$$

Furthermore, the least C above is equal to $||f||_{BLO(\mathcal{T})}$.

Proof. Suppose that $||f||_{BLO(\mathcal{T})} < \infty$. For any $E \in \bigcup_n \mathcal{T}_n$, there is $A_E \subset E$ of measure 0 such that if $y \in E \setminus A_E$, then $f(y) \geq \operatorname{essinf}_{x\in E} f(x)$. For any $y \in X\setminus\bigcup_E A_E$ and $\varepsilon > 0$, we find a set $E \in \bigcup_n \mathcal{T}_n$ such that $\mathcal{M}_{\mathcal{T}} f(y) \leq \frac{1}{\mu(E)}\int_E f\mathrm{d}\mu+\varepsilon$ and then

$$\mathcal{M}_{\mathcal{T}} f(y) - f(y) \leq \frac{1}{\mu(E)}\int_E f(x)\mathrm{d}\mu(x) - \operatorname*{ess\,inf}_{x\in E} f(x) + \varepsilon \leq ||f||_{BLO(\mathcal{T})} + \varepsilon.$$

Since ε was arbitrary and the set $X \setminus \bigcup_E A_E$ is of full measure, the condition (2.2) holds with $C = ||f||_{BLO(\mathcal{T})}$. To prove the reverse implication, suppose that a function f satisfies (2.2) and pick an arbitrary $E \in \bigcup_n \mathcal{T}_n$. For any $\varepsilon > 0$, we find a set $A \subset E$ of positive measure such that whenever $y \in A$, then

$$\begin{aligned}\frac{1}{\mu(E)}\int_E f(x)\mathrm{d}\mu(x) - \operatorname*{essinf}_{x\in E} f(x) &\leq \frac{1}{\mu(E)}\int_E f(x)\mathrm{d}\mu(x) - f(y) + \varepsilon\\ &\leq \mathcal{M}_{\mathcal{T}} f(y) - f(y) + \varepsilon \leq C + \varepsilon.\end{aligned}$$

This yields $||f||_{BLO(\mathcal{T})} \leq C$ and completes the proof. $\square$

We turn to the statement of the main result of the paper.

Theorem 2.5. *Let $(X, \mathcal{F}, \mu)$ be a probability space, equipped with an α-splitting tree $\mathcal{T}$. Then for any function f belonging to $BLO(\mathcal{T})$ and any number β satisfying*

$$0 < \beta < -\frac{\alpha\log\alpha}{(1-\alpha)||f||_{BLO(\mathcal{T})}}, \tag{2.3}$$

we have

$$\int_X \exp\left\{\beta\left(\mathcal{M}_{\mathcal{T}} f(x) - \int_X f(y)\,d\mu(y)\right)\right\} d\mu(x) \leq \frac{1-\alpha}{1-\alpha\exp\big(\beta(1-\alpha)/\alpha\big)}. \tag{2.4}$$

The constant on the right is the best possible for each n and each β. If $\beta \geq -\alpha\log\alpha/(1-\alpha)$, then there is a function f with $||f||_{BLO(\mathcal{T})} \leq 1$ for which the left-hand side of (2.4) is infinite.

In the light of the above example, this statement generalizes Theorem 1.1, which corresponds to $X = Q \subset \mathbb{R}^n$, $\mathcal{F} = \mathcal{B}(Q)$, $\mu = |\cdot|/|Q|$ and the 2^{-n}-splitting tree induced by dyadic cubes contained in Q.

3. On the method of proof

Now we will describe the technique which will be used to establish the announced inequalities. Throughout this section, we assume that $\alpha \in (0, 1/2]$ is a fixed constant. Distinguish the set

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \leq x + 1\}.$$

Let $\Phi, \Psi : \mathbb{R} \to \mathbb{R}$ be two given functions and assume we want to show that

$$\int_X \Phi\big(\mathcal{M}^n_{\mathcal{T}} f(x)\big)\, \mathrm{d}\mu(x) \leq \Psi\left(\int_X f(x) \mathrm{d}\mu(x)\right), \qquad n = 0, 1, 2, \ldots, \tag{3.1}$$

for any BLO function f with respect to an α-splitting tree $\mathcal{T}$, such that $||f||_{BLO(\mathcal{T})} \leq 1$. If Φ is sufficiently regular, say, nondecreasing and nonnegative, then (3.1) implies the corresponding bound with $\mathcal{M}^n_{\mathcal{T}}$ replaced by $\mathcal{M}_{\mathcal{T}}$ (by Lebegue's monotone convergence theorem; some other assumptions make the passage possible in view of Fatou's lemma). The key idea in the study of this problem is to construct a special function $B = B_{\alpha,\Phi,\Psi} : D \to \mathbb{R}$, which satisfies the following conditions.

1° We have $B(x, y) = B(x, x \vee y)$ for any $(x, y) \in D$.
2° We have $B(x, y) \geq \Phi(y)$ for any $(x, y) \in D$ satisfying $y \geq x$.
3° We have $B(x, x) \leq \Psi(x)$ for all $x \in \mathbb{R}$.
4° For any $(x, y) \in D$ with $x \leq y$ there exists $A = A(x, y) \in \mathbb{R}$ such that

$$B(x', y) \leq B(x, y) + A(x, y)(x' - x) \tag{3.2}$$

whenever $x' \in [y - 1, x + (1 - \alpha)/\alpha]$.

A few remarks concerning these conditions are in order. The condition 1° is a technical assumption which enables the efficient handling of the maximal operator. The conditions 2° and 3° are appropriate majorizations. The last condition imposes a concavity-type property on B; in particular, it has the following consequence.

Lemma 3.1. *Let (x, y) be a fixed point belonging to D, such that $x \leq y$, and let $n \geq 2$ be an arbitrary integer. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive numbers which sum up to 1, such that $\alpha_i \geq \alpha$ for each i. Assume further that $(x_1, y), (x_2, y), \ldots, (x_n, y) \in D$ satisfy $x = \sum_{i=1}^n \alpha_i x_i$. Then*

$$B(x, y) \geq \sum_{i=1}^n \alpha_i B(x_i, y). \tag{3.3}$$

Proof. Let us apply 4° to $x' = x_i$, $i = 1, 2, \ldots, n$, multiply both sides by α_i and finally sum the obtained inequalities. As the result of these operations, we get (3.3). Thus, all we need is to verify whether the requirement for x' appearing in 4° is fulfilled. The inequality $x' \geq y - 1$ is guaranteed by the condition $(x_i, y) \in D$. Furthermore, by the definition of D and the assumption $\alpha_i \geq \alpha$,

$$\begin{aligned} x = \sum_{i=1}^n \alpha_i x_i &\geq \alpha_i x_i + \sum_{j \neq i} \alpha_j (y - 1) \\ &= \alpha_i x_i + (1 - \alpha_i)(y - 1) \geq \alpha x_i + (1 - \alpha)(y - 1). \end{aligned}$$

Combining this with the inequality $x \leq y$ assumed in the statement, we obtain an inequality which is equivalent to $x_i \leq x + (1 - \alpha)/\alpha$. This completes the proof. □

We turn to the main result of this section.

Theorem 3.2. *Suppose that $(X, \mathcal{F}, \mu)$ is a probability space equipped with an α-splitting tree $\mathcal{T}$. If there is a function $B = B_{c,\alpha,\Phi,\Psi}$ satisfying 1°–4°, then* (3.1) *holds for any BLO function f with respect to $\mathcal{T}$ such that $||f||_{BLO(\mathcal{T})} \leq 1$.*

Proof. Let f be as in the statement. Define the associated sequence $(f_n)_{n\geq 0}$ of step function in the following manner: given a nonnegative integer n and $x \in X$, put

$$f_n(x) = \frac{1}{\mu(E)} \int_E f(t) \mathrm{d}\mu(t),$$

where E is the unique element of $\mathcal{T}_n$ which contains the point x. Now, let $E \in \mathcal{T}_n$ be an arbitrary set and let $E_1, E_2, \ldots, E_m$ be the elements of $\mathcal{T}_{n+1}$ whose union is E. Then we have

$$\frac{1}{\mu(E)} \int_E f_n(t) \mathrm{d}\mu(t) = \sum_{i=1}^m \frac{\mu(E_i)}{\mu(E)} \cdot \frac{1}{\mu(E_i)} \int_{E_i} f_{n+1}(t) \mathrm{d}\mu(t).$$

The inequality $||f||_{BLO(\mathcal{T})} \leq 1$ implies that the pair $(f_n, \mathcal{M}^n_{\mathcal{T}} f)$ takes values in D, see Lemma 2.4 above. These conditions, combined Lemma 3.1, yield the inequality

$$\int_E B(f_n(t), \mathcal{M}^n_{\mathcal{T}} f(t)) \mathrm{d}\mu(t) \geq \int_E B(f_{n+1}(t), \mathcal{M}^{n+1}_{\mathcal{T}} f(t)) \mathrm{d}\mu(t). \tag{3.4}$$

Indeed, we have $\mathcal{M}^{n+1}_{\mathcal{T}} f = \mathcal{M}^n_{\mathcal{T}} f \vee f_{n+1}$, so by 1°,

$$B(f_{n+1}(t), \mathcal{M}^{n+1}_{\mathcal{T}} f(t)) = B(f_{n+1}(t), \mathcal{M}^n_{\mathcal{T}} f(t)) \qquad \text{for } t \in E.$$

The functions f_n, $\mathcal{M}^n_{\mathcal{T}} f$ are constant on E; denote their values by x and y, respectively. Furthermore, f_{n+1} is constant on each E_j, $j = 1, 2, \ldots, n$; let x_j denote the corresponding value. Finally, put $\alpha_j = \mu(E_j)/\mu(E)$, which is a number not smaller that α, since $\mathcal{T}$ is α-splitting. Applying (3.3) with the above choice of parameters and integrating over E, we get (3.4). Summing over all $E \in \mathcal{T}_n$, we obtain

$$\int_X B(f_n(t), \mathcal{M}^n_{\mathcal{T}} f(t)) \mathrm{d}\mu(t) \geq \int_X B(f_{n+1}(t), \mathcal{M}^{n+1}_{\mathcal{T}} f(t)) \mathrm{d}\mu(t)$$

and therefore, by induction,

$$\int_X B(f_0(t), \mathcal{M}^0_{\mathcal{T}} f(t)) \mathrm{d}\mu(t) \geq \int_X B(f_n(t), \mathcal{M}^n_{\mathcal{T}} f(t)) \mathrm{d}\mu(t).$$

However, the left-hand side equals

$$B\left(\int_X f(t) \mathrm{d}\mu(t), \int_X f(t) \mathrm{d}\mu(t) \right),$$

and hence the application of 2° and 3° completes the proof of (3.1). $\square$

4. An exponential estimate

4.1. **Proof of** (2.4)**.** Now we will see how the methodology developed in the preceding section can be used to yield the exponential estimate of Theorem 2.5. Let $\alpha \in [0, 1/2]$ and an α-splitting tree $\mathcal{T}$ be fixed. We start from the observation that by homogeneity, it suffices to prove the assertion under the additional assumption $||f||_{BLO(\mathcal{T})} \leq 1$. Define the functions Φ, Ψ by

$$\Phi(x) = e^{\beta x}, \qquad \Psi(x) = \frac{1-\alpha}{1-\alpha e^{\beta(1-\alpha)/\alpha}} e^{\beta x}, \qquad x \in \mathbb{R}.$$

The Bellman function B corresponding to the inequality (2.4) is given as follows. For $(x,y)\in D$, let

$$B(x,y)=\big((x\vee y)-x\big)e^{\beta(x\vee y)}+\big(1+x-(x\vee y)\big)\Psi(x\vee y).$$

To establish the desired estimate, it suffices to check that B satisfies the conditions 1°–4° listed in the previous section. The property 1° follows directly from the definition. The inequality 2° is equivalent to

$$(1+x-y)e^{\beta y}\left[\frac{1-\alpha}{1-\alpha\exp(\beta(1-\alpha)/\alpha)}-1\right]\geq 0,$$

which follows directly from the trivial bound $\beta(1-\alpha)/\alpha\geq 0$ and the assumption $\beta<-\alpha\log\alpha/(1-\alpha)$. Both sides of the inequality in 3° are equal. Finally, to verify 4°, pick $(x,y)\in D$ with $x\leq y$ and take

$$A(x,y)=-e^{\beta(y)}+\Psi(y).$$

Then (3.2) can be rewritten in the equivalent form

$$\begin{aligned}F(x'):=&\big((x'\vee y)-x'\big)e^{\beta(x'\vee y)}+\big(1+x'-(x'\vee y)\big)\Psi(x'\vee y)\\&-(y-x')e^{\beta y}-(1+x'-y)\Psi(y)\leq 0,\end{aligned}$$

for $x'\in[y-1,x+(1-\alpha)/\alpha]$. On the interval $[y-1,y]$, the function F vanishes. On the other hand, for $x'\in[y,x+(1-\alpha)/\alpha]$, we have

$$F(x')=\Psi(x')-(y-x')e^{\beta y}-(1+x'-y)\Psi(y).$$

Let us extend F to the larger interval $[y-1,y+(1-\alpha)/\alpha]$, by the use of the above formula (for $x'\in(x+(1-\alpha)/\alpha,y+(1-\alpha)/\alpha]$). Then F is convex on $[y,y+(1-\alpha)/\alpha]$ and satisfies $F(y)=F(y+(1-\alpha)/\alpha)=0$, so it is nonpositive there and hence also on its whole domain. Thus, in view of the results of Section 2, the inequalities (1.4) and (2.4) follow.

4.2. **Sharpness.** To show that the constants appearing in (1.4) and (2.4) cannot be replaced by smaller numbers, we will construct appropriate examples. Let $\alpha\in(0,1/2]$ be a fixed splitting parameter. Assume that the underlying probability space $(X,\mathcal{F},\mu)$ is the interval $(0,1]$ with its Borel subsets and Lebesgue measure. Equip this triple with the tree $\mathcal{T}=(\mathcal{T}_0,\mathcal{T}_1,\mathcal{T}_2,\ldots)$ given by

$$\mathcal{T}_n=\big\{(0,\alpha^n],(\alpha^n,\alpha^{n-1}],\ldots,(\alpha,1]\big\},\qquad n=0,\,1,\,2,\,\ldots.$$

This tree is α-splitting: indeed, the only difference between $\mathcal{T}_n$ and $\mathcal{T}_{n+1}$ is that the set $(0,\alpha^n]\in\mathcal{T}_n$ is split into $(0,\alpha^{n+1}]$ and $(\alpha^{n+1},\alpha^n]$, and we have

$$\frac{\mu((0,\alpha^{n+1}])}{\mu((0,\alpha^n])}=\alpha,\qquad\frac{\mu((\alpha^{n+1},\alpha^n])}{\mu((0,\alpha^n])}=1-\alpha\geq\alpha.$$

Consider a function $f:X\to[0,\infty)$, defined by

$$f(x)=\sum_{n\geq 0}n(\alpha^{-1}-1)\chi_{(\alpha^{n+1},\alpha^n]}(x).$$

This function belongs to $BLO(\mathcal{T})$ and satisfies $||f||_{BLO(\mathcal{T})}\leq 1$. To see this, fix $x\in X$ and a set E satisfying $x\in E$ and $E\subset\mathcal{T}_n$ for some $n\geq 0$. Then we have two possibilities: either E is equal to $(\alpha^{k+1},\alpha^k]$ for some $k\leq n$ and then the average f_E is equal to $k(\alpha^{-1}-1)$, or $E=(0,\alpha^n]$ and then we compute that

$$f_E=\frac{1}{\alpha^n}\sum_{k\geq n}k(\alpha^{-1}-1)\cdot(\alpha^k-\alpha^{k+1})=n(\alpha^{-1}-1)+1.$$

This shows that

$$\mathcal{M}_{\mathcal{T}}f(x)=\sum_{n\geq 0}\big[n(\alpha^{-1}-1)+1\big]\chi_{(\alpha^{n+1},\alpha^n]}(x)=f(x)+1$$

and proves that actually we have the equality $||f||_{BLO(\mathcal{T})}=1$ (see Lemma 2.4). Now, for a fixed β satisfying (2.3), we derive that

$$\begin{aligned}\int_X &\exp\left\{\beta\left(\mathcal{M}_{\mathcal{T}}f(x)-\int_X f(y)\mathrm{d}\mu(y)\right)\right\}\mathrm{d}\mu(x)\\ &=\int_X\exp\big\{\beta(f(x)+1-f_X)\big\}\mathrm{d}\mu(x)\\ &=\int_X\exp\big\{\beta f(x)\big\}\mathrm{d}\mu(x)\\ &=\sum_{n\geq 0}\exp\big\{\beta n(\alpha^{-1}-1)\big\}(\alpha^n-\alpha^{n+1})\\ &=\frac{1-\alpha}{1-\alpha\exp\big\{\beta(1-\alpha)/\alpha\big\}}.\end{aligned}$$

This shows that equality can hold in (2.4). This also proves the final assertion of Theorem 2.5 (the right-hand side of (2.4) is infinite for sufficiently large β).

A few words about the sharpness of (1.4) are in order. The above example can be easily modified to work for any dyadic $Q\subset\mathbb{R}^n$ equipped with the lattice of its dyadic sub-cubes. In fact, applying an affine transformation if necessary, we may focus on the case $Q=(0,1]^n$. To find the right function, we take $\alpha=2^{-n}$ and exploit an appropriate "measure-preserving" transformation of (X,μ) above onto $((0,1]^n,|\cdot|)$. Namely, the role of $(0,\alpha^k]$ is played by $(0,2^{-k}]^n$, and the interval $(\alpha^k,\alpha^{k+1}]$ must be replaced by $(0,2^{-k-1}]^n\setminus(0,2^{-k}]^n$. This leads to

$$f(x)=\sum_{k\geq 0}k(2^n-1)\chi_{(0,2^{-k}]^n\setminus(0,2^{-k-1}]^n}(x),$$

which satisfies $||f||_{BLO^d((0,1]^n)}=1$ and

$$\int_{(0,1]^n}\exp\left\{\beta\left(\mathcal{M}^d_{(0,1]^n}f(x)-f_{[0,1]^n}\right)\right\}=\frac{2^n-1}{2^n-\exp\big(\beta(2^n-1)\big)}$$

for $\beta<n\log 2/(2^n-1)$. If $\beta\geq n\log 2/(2^n-1)$, then the integral on the left is infinite. This completes the proof of Theorem 1.1.

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