ON MARTINGALES WHOSE EXPONENTIAL PROCESSES SATISFY MUCKENHOUPT'S CONDITION A_1

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Abstract. Let X be a continuous-path uniformly integrable martingale such that its exponential process $\mathcal{E}(X)$ satisfies the probabilistic version of Muckenhoupt's condition A_1. We establish optimal upper bounds for the BMO norm of X and a class of related sharp exponential estimates.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, filtered by $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Throughout the paper, X will be an adapted uniformly integrable martingale with continuous trajectories, and $\langle X\rangle$ will denote the quadratic covariance process (or square bracket) of X. See e.g. Dellacherie and Meyer [1] for the necessary definitions. Let

$$\mathcal{E}(X) = \big(\exp(X_t - \langle X\rangle_t/2)\big)_{t\geq 0}$$

stand for the exponential local martingale induced by X. For $1 < p < \infty$, we say that $\mathcal{E}(X)$ satisfies Muckenhoupt's A_p condition (in short, $\mathcal{E}(X) \in A_p$), if

$$\sup_{t\geq 0} \left\| \mathcal{E}(X)_t \mathbb{E}\Big[\mathcal{E}(X)_\infty^{-1/(p-1)} \big| \mathcal{F}_t\Big]^{p-1} \right\|_\infty < \infty.$$

There is a version of this condition if we pass with p to 1. Namely, $\mathcal{E}(X)$ belongs to the class A_1, if

$$\sup_{t\geq 0} \left\| \mathcal{E}(X)_t \mathcal{E}(X)_\infty^{-1} \right\|_\infty < \infty.$$

The above supremum will be denoted by $||\mathcal{E}(X)||_{A_1}$ and called the A_1 constant of $\mathcal{E}(X)$. These A_p classes, introduced by Izumisawa and Kazamaki in [3], are probabilistic counterparts of the classical analytic A_p classes, defined by Muckenhoupt in [7] during the study of weighted inequalities for the Hardy-Littlewood maximal operator.

One of the objectives of this note is to study the interplay between the A_1 constant of $\mathcal{E}(X)$ and the BMO-norm of X. Recall that the martingale X is of bounded mean oscillation, if

$$||X||_{BMO} = \sup_{t\geq 0} \left\| \mathbb{E}\big[|X_\infty - X_t|^2 \,\big|\, \mathcal{F}_t\big]^{1/2} \right\|_\infty < \infty.$$

See Getoor and Sharpe [2], Kazamaki [6] for more details, and consult John and Nirenberg [4] for the original, analytic version of the BMO class.

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It is well known that X belongs to the class BMO if and only if its exponential process $\mathcal{E}(X)$ belongs to a class A_p for some $p>1$. See e.g. Kazamaki [5], [6]. On the other hand, using Hölder's inequality, we easily check that $A_p \subseteq A_q$ if $p \leq q$. Combining these two facts, we see that the condition $\mathcal{E}(X) \in A_1$ implies that $X \in BMO$, and one of our main results is the following sharp bound for $||X||_{BMO}$ in terms of $||\mathcal{E}(X)||_{A_1}$. Here and below, "log" stands for the natural logarithm.

Theorem 1.1. *For any uniformly integrable martingale X we have*

$$||X||_{BMO} \leq \left(2\log||\mathcal{E}(X)||_{A_1} + \frac{2}{||\mathcal{E}(X)||_{A_1}} - 1\right)^{1/2} \tag{1.1}$$

and the inequality is sharp.

The martingale version of the inequality of John and Nirenberg (see Getoor and Sharpe [2]) states that if X is of bounded mean oscillation and starts from 0, then $\mathbb{E}e^{\alpha X_\infty} < \infty$ for α belonging to some interval containing 0. Thus, in view of the above theorem, if $||\mathcal{E}(X)||_{A_1} < \infty$, then it is exponentially integrable in the previous sense. Our second result concerns the precise information on the set of admissible α's and the size of $\mathbb{E}e^{\alpha X_\infty}$. For the precise formulation, we need some extra notation. For any $\alpha < 1/4$ and $c \geq 1$, put

$$C(\alpha, c) = \frac{\lambda_+ - \lambda_-}{(\alpha - \lambda_-)c^{\alpha-\lambda_+} + (\lambda_+ - \alpha)c^{\alpha-\lambda_-}},$$

where $\lambda_\pm = (1 \pm \sqrt{1-4\alpha})/2$; for $\alpha = 1/4$ and $c \geq 1$, let

$$C(\alpha, c) = \frac{c^{1/4}}{1 + \log c^{1/4}}.$$

Finally, for $\alpha > 1/4$ and $c \geq 1$, define

$$C(\alpha, c) = c^{1/2-\alpha}\left[\frac{1-2\alpha}{\sqrt{4\alpha-1}}\sin\left(\frac{\sqrt{4\alpha-1}}{2}\log c\right) + \cos\left(\frac{\sqrt{4\alpha-1}}{2}\log c\right)\right]^{-1},$$

provided the expression in the square brackets is nonzero (and put $C(\alpha, c) = \infty$ otherwise).

Theorem 1.2. *Suppose that X is a uniformly integrable martingale with $X_0 = 0$. Let α_0 be the least $\alpha \in (1/4, \infty)$ satisfying*

$$\frac{2\alpha-1}{\sqrt{4\alpha-1}}\sin\left(\frac{\sqrt{4\alpha-1}}{2}\log||\mathcal{E}(X)||_{A_1}\right) = \cos\left(\frac{\sqrt{4\alpha-1}}{2}\log||\mathcal{E}(X)||_{A_1}\right) \tag{1.2}$$

(if $||\mathcal{E}(X)||_{A_1} = 1$, set $\alpha_0 = \infty$). Then for any $\alpha < \alpha_0$ we have

$$\mathbb{E}e^{\alpha X_\infty} \leq C(\alpha, ||\mathcal{E}(X)||_{A_1}) \tag{1.3}$$

and the inequality is sharp. If $\alpha \geq \alpha_0$, then the above exponential inequality does not hold with any finite constant C depending only on $||\mathcal{E}(X)||_{A_1}$.

As an interesting corollary, we obtain that if $\mathcal{E}(X)$ belongs to the class A_1, then $e^{X_\infty} \in L^{1/4}$ and the exponent $1/4$ cannot be enlarged. Higher integrability of e^{X_∞} implies the corresponding upper bound for the A_1 constant of $\mathcal{E}(X)$.

A few words about the organization of this note are in order. We establish Theorem 1.1 in the next section; Section 3 is devoted to the proof of Theorem 1.2.

2. On the BMO estimate

We start with rephrasing the condition A_1 in terms of the maximal function $\mathcal{E}(X)^*$ of $\mathcal{E}(X)$, given by $\mathcal{E}(X)^*_t = \sup_{0\leq s<t}\mathcal{E}(X)_s$, $0 \leq t \leq \infty$.

Lemma 2.1. *Let X be a martingale starting from 0 and let $c \geq 1$. The following conditions are equivalent.*

(i) X is uniformly integrable and satisfies $||\mathcal{E}(X)||_{A_1} \leq c$.

(ii) the pair $(\mathcal{E}(X), \mathcal{E}(X)^*)$ takes values in the cone $\{(y,z) \in \mathbb{R}^2_+ : y \leq z \leq cy\}$ with probability 1.

Proof. (i)⇒(ii) The process $\mathcal{E}(X)$ is a nonnegative local martingale, and thus it is a supermartingale. Hence, for any $s \leq t$ we have

$$\mathcal{E}(X)_s = \mathbb{E}\big[\mathcal{E}(X)_s|\mathcal{F}_t\big] \leq \mathbb{E}\big[c\mathcal{E}(X)_\infty|\mathcal{F}_t\big] \leq c\mathcal{E}(X)_t.$$

This implies $\mathcal{E}(X)^*_t \leq c\mathcal{E}(X)_t$ for any t, which is exactly what we need.

(ii)⇒(i) It will be proved in Lemma 2.2 below that if $\mathcal{E}(X)^* \leq c\mathcal{E}(X)$ for some $c > 0$, then X is bounded in L^2 and hence it is uniformly integrable. The condition $||\mathcal{E}(X)||_{A_1} \leq c$ is evident: we have $\mathcal{E}(X)_t \leq \mathcal{E}(X)^*_\infty \leq c\mathcal{E}(X)_\infty$ for any $t \geq 0$. □

Let c be a fixed number larger than 1. The key role in the proof of Theorem 1.1 is played by the function U, given on $\{(x,y,z) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ : y \leq z \leq cy\}$ by the formula

$$U(x,y,z) = x^2 + 2\log\left(\frac{cy}{z}\right) - \frac{2y}{z} + \frac{2}{c}.$$

Observe that the function $s \mapsto 2\log s - 2s$ is increasing on $(0,1]$, which implies

$$x^2 \leq U(x,y,z) \leq x^2 + 2\log c + \frac{2}{c} - 1. \tag{2.1}$$

The key property of U is the following.

Lemma 2.2. *Let X be a uniformly integrable martingale starting from 0 such that $||\mathcal{E}(X)||_{A_1} \leq c$. Then the process $\mathcal{U}^X = (U(X_t, \mathcal{E}(X)_t, \mathcal{E}(X)^*_t)_{t\geq0}$ is a uniformly integrable martingale.*

Proof. First we show that $\mathcal{U}^X$ is a local martingale, using Itô's formula. We have $U_{xy} = 0$, $\mathrm{d}\langle\mathcal{E}(X)\rangle_t = \mathcal{E}(X)^2_t \mathrm{d}\langle X\rangle_t$ and $U_{xx} + y^2U_{yy} = 0$, which implies that the integral with respect to $\langle X\rangle$ vanishes. Similarly, we have $U_z(x,z,z) = 0$, which gives

$$\int_{0+}^t U_z(X_s, \mathcal{E}(X)_s, \mathcal{E}(X)^*_s)\,\mathrm{d}\mathcal{E}(X)^*_s = 0,$$

since the process $\mathcal{E}(X)^*$ increases for t lying in the set $\{s : \mathcal{E}(X)_s = \mathcal{E}(X)^*_s\}$. This yields the local martingale property of $\mathcal{U}^X$. Denoting the localizing sequence by $(\tau_n)_{n\geq1}$, we obtain, by the left inequality in (2.1),

$$\mathbb{E}X^2_{\tau_n\wedge t} \leq \mathbb{E}\mathcal{U}^X_{\tau_n\wedge t} = \mathcal{U}^X_0 = U(0,1,1), \qquad t \geq 0.$$

Since n and t were arbitrary, Doob's maximal inequality implies that $X^* = \sup_{t\geq0}|X_t|$ is in L^2. Therefore, using the upper bound in (2.1), we see that $\mathcal{U}^X$ is majorized by an integrable random variable. This yields the claim. □

Proof of Theorem 1.1. Fix a nonnegative number t. Of course, the process $Y = (X_{u\vee t} - X_t)_{u\geq 0}$ is a uniformly integrable martingale starting from 0. Furthermore, we have $\mathcal{E}(Y)_u = \mathcal{E}(X)_{u\vee t}/\mathcal{E}(X)_t$, so $||\mathcal{E}(Y)||_{A_1} \leq ||\mathcal{E}(X)||_{A_1}$. Applying Lemma 2.2 to the process Y and $c := ||\mathcal{E}(X)||_{A_1}$ we obtain, by virtue of (2.1),

$$\begin{aligned} \mathbb{E}\big[|X_\infty - X_t|^2|\mathcal{F}_t\big] &= \mathbb{E}\big[Y_\infty^2|\mathcal{F}_t\big] \leq \mathbb{E}\big[\mathcal{U}_\infty^Y|\mathcal{F}_t\big] = \mathcal{U}_t^Y \\ &= 2\log||\mathcal{E}(X)||_{A_1} + \frac{2}{||\mathcal{E}(X)||_{A_1}} - 1. \end{aligned} \tag{2.2}$$

This completes the proof of (1.1), since t was arbitrary. To see that this bound is sharp, fix $c \geq 1$, let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion and consider the stopping time $\tau = \inf\{t \geq 0 : \mathcal{E}(B)_t^* = c\mathcal{E}(B)_t\}$. If we repeat the reasoning from (2.2), with $t = 0$ and $X = Y = B^\tau$, we see that the inequality becomes an equality. Hence both sides of (1.1) are equal and the proof is finished. □

3. Exponential estimates

As we have seen in the statement of Theorem 1.2, the optimal upper bounds for $\mathbb{E}e^{\alpha X_\infty}$ are given by three different formulas, depending on the value of c and α. To enable the unified treatment of these inequalities, we will first prove the following.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, $c > 1$ be fixed. Suppose that there exists a function $f : [c^{-1}, 1] \to [1,\infty)$ of class C^2 which satisfies the equalities*

$$\alpha^2 f(s) + 2\alpha s f'(s) + s^2 f''(s) = 0, \qquad s \in (c^{-1}, 1), \tag{3.1}$$

$$f'(1-) = 0 \tag{3.2}$$

and

$$f(c^{-1}) = 1. \tag{3.3}$$

Then for any uniformly integrable martingale X with $X_0 = 0$ and $||\mathcal{E}(X)||_{A_1} \leq c$ we have

$$\mathbb{E}\exp(\alpha X_\infty) \leq f(1). \tag{3.4}$$

The inequality is sharp.

Proof. The reasoning is similar to that appearing in the proof of Theorem 1.1 and rests on the existence of a certain special function. Namely, let us introduce $U : \{(x, y, z) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ : y \leq z \leq cy\} \to \mathbb{R}$ given by

$$U(x, y, z) = e^{\alpha x} f(y/z).$$

By Itô's formula, the process $\mathrm{U}^X = (U(X_t, \mathcal{E}(X)_t, \mathcal{E}(X)_t^*))_{t\geq 0}$ is a local martingale. Indeed, the equation (3.1) implies that the integral with respect to $\langle X\rangle$ vanishes; moreover, (3.2) enforces $U_z(x, z, z) = 0$ and hence the integral with respect to $\mathcal{E}(X)^*$ is zero. Let $(\tau_n)_{n\geq 1}$ be the localizing sequence of U^X. Since f takes values in $[1,\infty)$, we have $\mathrm{U}^X \geq e^{\alpha X}$ and hence, for any n and t,

$$\mathbb{E}\exp(\alpha X_{\tau_n\wedge t}) \leq \mathbb{E}\mathrm{U}^X_{\tau_n\wedge t} = \mathrm{U}^X_0 = f(1). \tag{3.5}$$

Letting n and t to infinity we get (3.4), by virtue of Fatou's lemma. To see that this inequality is sharp, observe that $(\exp(\alpha X_{\tau_n\wedge t}/2))_{t\geq 0}$ is a nonnegative submartingale. Consequently, by Doob's maximal inequality in L^2, if we let $t \to \infty$ and

$n \to \infty$ in (3.5), we get that $\sup_{t\geq 0} \exp(\alpha X_t)$ is integrable and thus U^X is a uniformly integrable martingale. Now, as previously, we take X to be a standard Brownian motion $B = (B_t)_{t\geq 0}$ stopped at the time $\tau = \inf\{t \geq 0 : \mathcal{E}(B)^*_t = c\mathcal{E}(B)_t\}$. Then $||\mathcal{E}(B^\tau)||_{A_1} = c$ and, since $f(1/c) = 1$,

$$\mathbb{E}\exp(\alpha B_\tau) = \mathbb{E}\mathrm{U}^{B^\tau}_\infty = \mathrm{U}^{B^\tau}_0 = f(1).$$

This completes the proof. □

We turn to the proof of Theorem 1.2 and consider the cases $\alpha \leq 1/4$ and $\alpha > 1/4$ separately. We may assume that $c := ||\mathcal{E}(X)||_{A_1} > 1$, since if the A_1 constant of $\mathcal{E}(X)$ is equal to 1, then X is zero almost surely and the claim is obvious.

Proof of Theorem 1.2 for $\alpha \leq 1/4$, $c > 1$. The function $C(\cdot,\cdot)$ is continuous on the set $(-\infty, \frac{1}{4}] \times [1,\infty)$. Thus, by Lebesgue's monotone convergence theorem, we may restrict ourselves to α's which are strictly smaller than 1/4. It is not difficult to determine a function f which satisfies (3.1), (3.2) and (3.3): we have $f \equiv 1$ for $\alpha = 0$ and

$$f(s) = \frac{(\alpha-\lambda_-)s^{\lambda_+-\alpha}}{(\alpha-\lambda_-)c^{\alpha-\lambda_+} + (\lambda_+-\alpha)c^{\alpha-\lambda_-}} + \frac{(\lambda_+-\alpha)s^{\lambda_--\alpha}}{(\alpha-\lambda_-)c^{\alpha-\lambda_+} + (\lambda_+-\alpha)c^{\alpha-\lambda_-}}$$

for $\alpha \neq 0$ (here, as in the statement of Theorem 1.2, $\lambda_\pm = (1 \pm \sqrt{1-4\alpha})/2$). Observe that f takes values in $[1,\infty)$. This is clear for $\alpha = 0$. For the remaining values of the parameter α, we compute that

$$f'(s) = \frac{(\alpha-\lambda_-)(\lambda_+-\alpha)(s^{\lambda_+-\alpha-1} - s^{\lambda_--\alpha-1})}{(\alpha-\lambda_-)c^{\alpha-\lambda_+} + (\lambda_+-\alpha)c^{\alpha-\lambda_-}}.$$

It suffices to note that $\alpha - \lambda_- = \sqrt{1-4\alpha} + 2\alpha - 1 < 0$, $\lambda_+ - \alpha > 1 - \alpha > 0$, $s^{\lambda_+-\alpha-1} - s^{\lambda_--\alpha-1} < 0$ for $s \in (0,1)$ and

$$\begin{aligned}(\alpha-\lambda_-)c^{\alpha-\lambda_+} + (\lambda_+-\alpha)c^{\alpha-\lambda_-} &= c^{\alpha-\lambda_-}\left[(\alpha-\lambda_-)c^{\lambda_--\lambda_+} + (\lambda_+-\alpha)\right]\\ &\geq c^{\alpha-\lambda_-}\left[(\alpha-\lambda_-) + (\lambda_+-\alpha)\right] > 0.\end{aligned}$$

This shows that f is increasing on $(1/c, 1)$ and hence $f \geq f(1/c) = 1$. Thus, by Theorem 3.1, the inequality (1.3) holds true and the constant $C(\alpha, ||\mathcal{E}(X)||_{A_1})$ is the best possible. □

Proof of Theorem 1.2 for $\alpha > 1/4$, $c > 1$. First we will show that

$$\frac{\sqrt{4\alpha_0-1}}{2}\log c < \pi, \tag{3.6}$$

or, equivalently, $\alpha_0 < \frac{1}{4} + \frac{\pi^2}{\log^2 c}$. To do this, note that if we let $\alpha \to 1/4$ in (1.2), then the left-hand side tends to $-\log c/4$ and the right-hand side converges to 1; similarly, if we let $\alpha \to \frac{1}{4} + \frac{\pi^2}{\log^2 c}$, then the left-hand side of (1.2) converges to 0 and the right-hand side approaches -1. Thus (3.6) follows from Darboux property.

Now, suppose that $\alpha < \alpha_0$. It is easy to find a function which satisfies the differential equation (3.1) and the condition (3.2): let $F : [c^{-1}, 1] \to \mathbb{R}$ be given by

$$F(s) = s^{1/2-\alpha}\left[\frac{2\alpha-1}{\sqrt{4\alpha-1}}\sin\left(\frac{\sqrt{4\alpha-1}}{2}\log s\right) + \cos\left(\frac{\sqrt{4\alpha-1}}{2}\log s\right)\right].$$

The key fact is that F takes positive values. Indeed, we have $F(c^{-1})>0$ in view of (1.2); furthermore, an easy calculation shows that

$$F'(s)=-\frac{2\alpha^2}{\sqrt{4\alpha-1}}s^{-1/2-\alpha}\sin\left(\frac{\sqrt{4\alpha-1}}{2}\log s\right),\qquad s\in(c^{-1},1).$$

This is nonnegative, because

$$0>\frac{\sqrt{4\alpha-1}}{2}\log s>\frac{\sqrt{4\alpha_0-1}}{2}\log c^{-1}>-\pi,$$

where the latter passage is due to (3.6). Therefore the function $f(s)=F(s)/F(c^{-1})$, $s\in[c^{-1},1]$, satisfies (3.1), (3.2), (3.3) and takes values in $[1,\infty)$. An application of Theorem 3.1 gives the assertion.

Finally, suppose that $\alpha\geq\alpha_0$ and pick $\alpha_1<\alpha_0$. By Hölder's inequality, we have

$$\mathbb{E}e^{\alpha X_\infty}\geq\left[\mathbb{E}e^{\alpha_1 X_\infty}\right]^{\alpha/\alpha_1}$$

and by the appropriate choice of X, the right-hand side can be made equal to $C(\alpha_1,c)^{\alpha/\alpha_1}$. It suffices to note that $C(\alpha_1,c)\to\infty$ as $\alpha_1\uparrow\alpha$; this proves that the inequality (1.3) does not hold with any finite constant. The proof is complete. □

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