PROBLEMS WAS (ISA) – 2

1. Prove that in the space $\mathcal{X} = C = C([0,1], \mathbb{R})$ with metric $||x - y|| = \sup_{t\in[0,1]} |x(t) - y(t)|$ we have $\sigma(\mathcal{C}) = \mathcal{B}(C)$, where $\mathcal{C}$ denotes the set of cylinders in C.
Hint: C is separable.

2. Show that in the space $\mathcal{X} = \mathbb{R}^{\mathbb{R}_+}$ the sets $\{x \in \mathcal{X} : x \text{ is non} - \text{decreasing}\}$, $\{x \in \mathcal{X} : \sup |x(t)| \leq 1\}$ do not belong to $\sigma(\mathcal{C})$.
Hint: Prove first that if $A \in \sigma(\mathcal{C})$, then there exists a set $Z \subset \mathbb{R}_+$ at most countable, such that from the fact that $x \in A$ and $\forall_{t\in Z} x(t) = y(t)$ it follows that $y \in A$.

3. $(W_t)_{t\in\mathbb{R}_+}$ is a Wiener process.
a) Define $\overline{W}_0 = 0$, $\overline{W}_t = tW_{1/t}$ for $t > 0$. Prove that $\overline{W}$ is a Wiener process (time inversion).
b) Prove that $\lim_{t\to\infty} \frac{W_t}{t} = 0$ a.s.
Hint for a): The event $\{\lim_{t\searrow 0} \overline{W}_t = 0\}$ has the form $\{\overline{W} \in \Gamma\}$, where $\Gamma \subset \sigma(\mathcal{C})$ in the space of continuous functions on $(0, \infty)$.

4. T is an arbitrary non-empty set, and for each $t \in T$, μ_t is a probability measure on $\mathbb{R}$. Prove that there exists a probability space and a random function $(X_t)_{t\in T}$ such that X_t has distribution μ_t, $t \in T$, and the random variables $\{X_t\}_{t\in T}$ are independent.

5. a)Show that the paths of the Wiener process W satisfy locally (i.e., on each interval $[0, a]$) the Hölder condition with any exponent $\alpha \in (0, 1/2)$.
b)Prove that the paths of W do not satisfy the Hölder condition with exponent $\alpha = 1/2$; more precisely,
$P(\exists_{[a,b]\subset\mathbb{R}_+} W_\cdot$ satisfies the Hölder cond. with exponent $1/2$ on $[a, b]) = 0$.
Hint for b): For a fixed interval $[a, b]$ define $t_{n,k} = a + (b - a)k/n$, $k = 0, 1, \ldots, n$, and consider $W_{t_{n,k+1}} - W_{t_{n,k}}$.