Weighted Inequalities, Homework I. Deadline: 18.04.2019

1. Prove that for any integrable function f on $\mathbb{R}^d$ satisfying $\|f\|_{L^1(\mathbb{R}^d)} \neq 0$ and any $\alpha < 0$, the weight $w = (Mf)^\alpha$ belongs to A_∞.

2. Let f, g be two nonnegative functions on $\mathbb{R}^d$. Suppose that for some $0 < p_0 < \infty$ and all $w \in A_\infty$ we have the estimate $\|g\|_{L^{p_0}(w)} \leq C\|f\|_{L^{p_0}(w)}$, where C does not depend on w. Prove that for all $0 < p < \infty$ and any $w \in A_\infty$ we have $\|g\|_{L^p(w)} \leq K_{p,C}\|f\|_{L^p(w)}$ for some constant $K_{p,C}$ depending only on the parameters indicated.

3. Let T be a linear operator satisfying

$$\|Tf\|_{L^1(w)} \leq c_{p,d,T}[w]_{A_p}\|Mf\|_{L^1(w)}$$

for some $1 < p < \infty$ and any A_p weight w. Prove that there is a constant K such that for any $1 < p$, $r < \infty$ and any positive function u we have

$$\left\|\frac{Tf}{M_r u}\right\|_{L^p(M_r u)} \leq Kp\left\|\frac{Mf}{M_r u}\right\|_{L^p(M_r u)},$$

where $M_r w = M(w^r)^{1/r}$.

Hints: (i) Use duality: $\left\|\frac{Tf}{M_r u}\right\|_{L^p(M_r u)} = \sup\int_{\mathbb{R}^d}|Tf|h$, *supremum over an appropriate class.*
(ii) Use Rubio de Francia algorithm to construct an A_1 *weight* $R(h)(M_r w)^{1/p'}$.
(iii) What can you say about $[R(h)]_{A_p}$*? (Factorization of weights . . .).*
(iv) Put all the above facts together.

4. Prove that for any $1 < p < \infty$ there is a constant $c_{p,d}$ depending only on the parameters indicated such that the following holds. If w is an A_p weight on $\mathbb{R}^d$, $Q \subset \mathbb{R}^d$ is a dyadic cube and E is an arbitrary subset of Q, then

$$\frac{|E|}{|Q|} < e^{-c_{p,d}[w]_{A_p}} \quad \Rightarrow \quad \frac{w(E)}{w(Q)} < \frac{1}{100}.$$

Hints. Take a look at the implication (c)⇒(d) in Theorem 2.11. Apply the second part of Exercise 13 in Chapter 2 and combine it with Exercise 11.