Measure Theory - Problems for Test 4

1. A Borel set $E \subseteq \mathbb{R}^2$ has the property that each point from $[0,1]^2$ is a density point for E. Show that the Lebesgue measure of E is not smaller than 1.

2. Determine whether there exists a set $E \subset \mathbb{R}$ such that the set of density points of E is equal to $\mathbb{R} \setminus \{0\}$.

3. For $n \geq 1$, let μ_n be the following measure on $\mathbb{R}$:
(i) with density $n^2 \chi_{[n,2n]}$ (with respect to Lebesgue measure),
(ii) concentrated on the points $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n^3+5n+1}{n}\right\}$, with $\mu_n(\{k/n\}) = \frac{k^3+1}{n^4+4}$.
Does $(\mu_n)_{n\geq 1}$ converge weakly? If so, determine the limit measure.

4. Let X be a countable set and let τ be a topology on X with the following property. For any sequence $(\mu_n)_{n\geq 1}$ of measures on X, $(\mu_n)_{n\geq 1}$ converges weakly to μ if and only if for any $x \in X$, $\mu_n(\{x\}) \to \mu(\{x\})$. Prove that the topology is discrete.

5. Construct a subset of $\mathbb{R}$ of Hausdorff dimension 1 that has zero Lebesgue measure.
Hint: Modify Cantor set.

6. Derive the Hausdorff dimension of the ellipse $\{(x,y) \in \mathbb{R}^2 : x^2 + 4y^2 = 4\}$.