Measure Theory - Problems for Test 3

1. Let μ be a signed Borel measure on the space $(\mathbb{R}, \mathcal{F})$. Prove that for any Borel set $E \subset \mathbb{R}$,

$$|\mu|(E) = \sup \left\{ \sum_{k=1}^{\infty} |\mu(E_k)| \ : \ (E_k)_{k\geq 1} \text{ pairwise disjoint}, \bigcup_{k\geq 1} E_k \subset E \right\}.$$

2. Let μ be a signed Borel measure on $[0,1]$ and let $|\cdot|$ be a Lebesgue measure. Suppose that for any $|\cdot|$-summable function $f : [0,1] \to \mathbb{R} \cup \{\infty\}$ we have $\int_{[0,1]} f \mathrm{d}\mu < \infty$. Prove that there is a finite number M such that $|\mu|(E) \leq M|E|$ for any Borel subset E of $[0,1]$.

3. Let $(X, \mathcal{M})$ be a measurable space and suppose that μ, ν are real measures. Prove that $(\mu + \nu)^+ \leq \mu^+ + \nu^+$.

4. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let f, g be two measurable functions on X such that the set functions ν, λ given by

$$\nu(E) = \int_E f \mathrm{d}\mu, \qquad \lambda(E) = \int_E g \mathrm{d}\mu$$

are probability measures. Prove that

$$\sup_{A \in \mathcal{F}} \big|\nu(A) - \lambda(A)\big| = \frac{1}{2} \int_X |f - g| \mathrm{d}|\mu|.$$

5. Let U be a bounded open set in the plane. For any such set, let

$$r(U) = \sup\{r : U \text{ contains a disk of radius } r\}.$$

Define a sequence of closed disks as follows: D_1 is any closed disk in U with radius at least $\frac{1}{2}r(U)$; if D_1, D_2, ..., D_n have been defined, let $U_n = U \setminus (D_1 \cup \ldots \cup D_n)$ and let D_{n+1} be any disk contained in U_n with radius at least $\frac{1}{2}r(U_n)$. Prove that there is an absolute c (i.e., not depending on U or the choice of the disks) such that

$$\sum_{k=1}^{\infty} \mathrm{Area}\,(D_k) \ \geq c \, \mathrm{Area}(U).$$

6. Let μ be Lebesgue measure on $\mathbb{R}$. For any μ-locally integrable function f we define the associated minimal operator $\mathcal{M}f$ by

$$\mathcal{M}f(x) = \inf_I \frac{1}{\mu(I)} \int_I |f| \mathrm{d}\mu,$$

where the infimum is taken over all intervals I which contain x. Prove that for any $t > 0$,

$$\mu(\{x : \mathcal{M}f(x) < t\}) \geq \frac{1}{2t} \int_{\{\mathcal{M}f < t\}} |f| \mathrm{d}\mu.$$