**Measure Theory - Problems for Test 2**

**1.** Let $E$ be a set which is not Lebesgue measurable. Define

$$f(x) = \begin{cases} x & \text{if } x \in E, \\ -x & \text{if } x \notin E. \end{cases}$$

Can $f$ be Lebesgue measurable?

**2.** Let $\mu$ be a Radon measure on $\mathbb{R}$ and let $f$ be a $\mu$-measurable, nonnegative function on $\mathbb{R}$. Prove that there is a $\mu$-measurable subset $E \subset \mathbb{R}$ with $0 < \mu(E) < \infty$ such that for all $x \in E$,

$$f(x) \geq \frac{1}{2\mu(E)} \int_E f(y) \mathrm{d}\mu(y).$$

**3.** For $t > 0$ and $x \in \mathbb{R}$, define

$$g(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$$

and $h(t,x) = \frac{\partial g}{\partial t}$. Given $a > 0$, prove that

$$\int_{\mathbb{R}} \left( \int_a^\infty h(t,x) \mathrm{d}t \right) \mathrm{d}x = -1, \qquad \int_a^\infty \left( \int_{\mathbb{R}} h(t,x) \mathrm{d}x \right) \mathrm{d}t = 0$$

and conclude that

$$\int_{(a,\infty] \times \mathbb{R}} |h(t,x)| \mathrm{d}t\mathrm{d}x = \infty.$$

**Hint:** For a fixed $t > 0$, $g$ is a density of a Gaussian distribution, so $\int_{\mathbb{R}} g(t,x)\mathrm{d}x = 1$; furthermore, we have $\frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2 g}{\partial x^2}$.

**4.** Let $f$ be a continuous and compactly supported function on $\mathbb{R}$. Prove that for any Lebesgue measurable function $g : \mathbb{R} \to \mathbb{R}$ of period 1 we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) g(nx) \mathrm{d}x = \int_{\mathbb{R}} f(x) \mathrm{d}x \int_0^1 g(x) \mathrm{d}x.$$

**5.** Let $(X, \mu)$ be a finite measurable space and let $f : X \to [-2012, \infty)$ be a $\mu$-measurable function. Prove that

$$\lim_{n\to\infty} \int_X \left(1 + \frac{f(x)}{n}\right)^n \mathrm{d}x = \int_X e^{f(x)} \mathrm{d}\mu(x).$$

**6.** Let $\mu$ be a Radon measure on $\mathbb{R}$, suppose that $A$ is a $\mu$-measurable subset of $[a,b]$ and let $h$ be a positive number. Prove that

$$\frac{1}{2h} \int_a^b \mu\big(A \cap (x-h, x+h)\big) \mathrm{d}x \leq \mu(A).$$

**7.** A sequence $(f_n)_{n\geq 1}$ converges almost uniformly to $f$. Prove that $(\sqrt{|f_n|})_{n\geq 1}$ converges almost uniformly to $\sqrt{|f|}$.

**8.** A sequence $(f_n)_{n\geq 1}$ of $\mu$-measurable functions on $X$ satisfies $\int_X |f_n| \mathrm{d}\mu \to 0$.
(i) Does this imply that $(f_n)_{n\geq 1}$ converges to 0 in measure?
(ii) Does this imply that $(f_n)_{n\geq 1}$ converges to 0 almost uniformly?