Measure Theory - Problems for Test 1

1. A collection $\mathcal{C}$ of subsets of X is an algebra with the following property: if $E_n \in \mathcal{C}$ and $E_n \subseteq E_{n+1}$, $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$. Prove that $\mathcal{C}$ is a σ-algebra.

2. Suppose $\mu_i^* : 2^{\mathbb{R}} \to [0, 1]$, $i = 1, 2$, are outer measures.
(i) Prove that $\mu^* = \max(\mu_1^*; \mu_2^*)$ is an outer measure.
(ii) Suppose that μ_i^* are regular. Is μ^* regular?
(iii) Suppose that μ_i^* are Borel. Is μ^* Borel?
Hint: Use Dirac measures.

3. We say that a set $E \subseteq \mathbb{R}$ has an infinite density point, if there is an uncountable number of elements of E outside any bounded interval. Define μ^* on $2^{\mathbb{R}}$ by

$$\mu^*(E) = \begin{cases} 0 & \text{if } E \text{ is at most countable,} \\ 1 & \text{if } E \text{ is uncountable, but does not have an infinite density point,} \\ \infty & \text{if } E \text{ has an infinite density point.} \end{cases}$$

Prove that μ^* is an outer measure and identify the σ-algebra of μ^*-measurable sets.

4. Assume that μ^* is an outer measure on 2^X and pick $A \subseteq C \subseteq X$. Show that if a set B is μ^*-measurable and satisfies $A \subseteq B$ and $\mu^*(A) = \mu^*(B)$, then $\mu^*(C) = \mu^*(B \cup C)$.

5. Consider the outer measure $\mu^* = \sum_{n \geq 1} \delta_{1/n}$ on $2^{\mathbb{R}}$.
(i) Is it true that for any Borel set $A \subset \mathbb{R}$ we have

$$\mu^*(A) = \inf\{\mu^*(U) : A \subset U,\ U \text{ open}\}?$$

(ii) Is it true that for any Borel set $A \subset \mathbb{R}$ we have

$$\mu^*(A) = \sup\{\mu^*(F) : F \subset A,\ F \text{ closed}\}?$$

6. Assume that μ is a Borel measure on $[0, 1]$ and let $|\cdot|$ be the Lebesgue measure on $[0, 1]$. Suppose that for any Borel set $A \subset [0, 1]$ with $|A| = 1/2$ we have $\mu(A) = 1/2$. Prove that $\mu = |\cdot|$.

7. For $k \geq 1$, let $A_k = \{0, k, 2k, \ldots\}$. Show that there is no measure μ on $\mathbb{N}$ satisfying $\mu(A_k) = 1/k$ for all $k \geq 1$.
Hint: Use probability theory and Borel-Cantelli lemma.

8. Let μ be the Lebesgue measure on $\mathbb{R}$ and let E be a Borel subset of $\mathbb{R}$ such that

$$\mu(E \setminus (E + x)) = 0 \qquad \text{for any } x \in \mathbb{R},$$

where $E + x = \{z + x : z \in E\}$. Prove that $\mu(E) = 0$ or $\mu(\mathbb{R} \setminus E) = 0$.