Measure Theory, some additional exercises

1. Let (μ_k) be a sequence of outer measures on $\mathbb{R}^d$ such that $\mu_{k+1}(E) \geq \mu_k(E)$ for each measurable set E. Define μ by $\mu(E) = \lim_{k\to\infty} \mu_k(E)$.
(a) Prove that μ is an outer measure.
(b) Suppose that μ_k are Borel regular. Does this imply that μ is Borel regular?
(c) Suppose that μ_k are Radon. Does this imply that μ is a Radon measure?

2. Prove that a function defined on $\mathbb{R}^n$, that is continuous everywhere except for a set of Lebesgue measure zero, is a Lebesgue measurable function.

3. Let f be a function defined on $\mathbb{R}^2$ that is continuous in each variable separately. Prove that f is Lebesgue measurable.

4. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose that $(f_j)_{j\geq 1}$ and f are measurable functions. Prove that $f_j \to f$ in measure if and only if each subsequence of (f_j) has a subsequence that converges to f μ-almost everywhere.

5. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let f, f_k, $k \geq 1$ be measurable functions with

$$\lim_{k\to\infty} f_k = f \qquad \mu\text{-almost everywhere.}$$

Prove that there are measurable sets E_0, E_1, E_2, $\ldots$ such that $\mu(E_0) = 0$, $X = \bigcup_{j=0}^{\infty} E_j$ and $f_k \to f$ uniformly on each E_j, $j = 1, 2, \ldots$.

6. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Prove that there is a function $f \in L^1(X, \mathcal{M}, \mu)$ such that $0 < f < 1$ μ-almost everywhere on X.

7. Let $f \in L^p(\mathbb{R}^n)$. Prove that if $1 \leq p < \infty$, then

$$\lim_{|h|\to 0} ||f(\cdot + h) - f||_p = 0.$$

Show that this result fails when $p = \infty$.
Remark: $f(\cdot + h)$ denotes the function $x \mapsto f(x + h)$.

8. Suppose that μ and ν are σ-finite measures on $(X, \mathcal{M})$ such that $\mu \ll \nu$ and $\nu \ll \mu$. Prove that $D_\mu \nu \neq 0$ and $D_\nu \mu = 1/D_\mu \nu$ almost everywhere (with respect to any of the measures μ, ν).

9. Let f and g be integrable functions on a measure space $(X, \mathcal{M}, \mu)$ with the property that

$$\mu(\{f > t\} \Delta \{g > t\}) = 0$$

for almost all $t \in \mathbb{R}$ (with respect to Lebesgue measure). Here $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B. Prove that $f = g$ μ-almost everywhere.

10. Let f be a Lebesgue measurable function on $[0, 1]$ and let $Q = [0, 1] \times [0, 1]$.
(a) Show that the function F given by $F(x, y) = f(x) - f(y)$ is measurable with respect to Lebesgue measure on $\mathbb{R}^2$.
(b) Prove that if $F \in L^1(Q)$, then $f \in L^1(0, 1)$.

11. Let μ be a measure on $[0, 1]$, given by $\mathrm{d}\mu = \frac{1}{x}\mathrm{d}x$.
(a) Suppose that f, f_1, f_2, $\ldots$ are functions on $[0, 1]$ such that $(f_n)_{n\geq 1}$ converges in μ-measure to f. Does this imply that (f_n) converges in Lebesgue measure to f?
(b) Let $f_n = \chi_{[0,1/n]}$, $n = 1, 2, \ldots$. Does $(f_n)_{n\geq 1}$ converge μ-almost everywhere, in μ-measure, μ-almost uniformly?
(c)* Derive $\lim_{n\to\infty} \int_0^1 \sin(nx)\mathrm{d}\mu$. Hint: use probability theory and characteristic functions.

12. Let $E \subset \mathbb{R}^d$ be a Borel set. Prove that $\dim_H \left(E \times [0, 1]\right) = \dim_H E + 1$.

13. Let $\mathcal{C}$ denote the usual Cantor set. Compute the Hausdorff dimension of the set

$$\{(x, y) \in \mathcal{C} \times \mathbb{R} : x|y| \leq 1\}.$$