Exercises in Measure Theory - 8+9

1. Let μ be the Lebesgue measure on the unit ball B of $\mathbb{R}^n$ and suppose that $S \subset B$ satisfies $\mu(B \setminus S) = 0$. Assume that ν is a Radon measure supported on $B \setminus S$. Prove that for μ-almost all $x \in B$, there is $\delta > 0$ such that $\nu(B(x,r)) \leq r^n$ for all $r \in (0,\delta)$.

2. Let μ be a Radon measure on $\mathbb{R}^n$. For any μ-locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, we define the Hardy-Littlewood maximal function $M_\mu f$ by

$$M_\mu f(x) = \sup \left\{ \frac{1}{\mu(B)} \int_B |f(y)| \mathrm{d}\mu(y) \right\},$$

where the supremum is taken over all balls B which contain x. Similarly, we define the centered maximal function $M_\mu^c f$, but the supremum is taken over all balls B centered at x.

(i) Let μ be the Lebesgue measure in $\mathbb{R}$. Compute $M_\mu \chi_{[-1,1]}$.

(ii) Let μ be the Lebesgue measure in $\mathbb{R}^n$. Prove that for any f as above,

$$\mu(\{x \in \mathbb{R}^n : M_\mu f(x) > 1\}) \leq C \int_{\mathbb{R}^n} |f(x)| \mathrm{d}\mu(x),$$

where $C = 5^n$.

(iii) Does the assertion of (ii) hold for general measures?

(iv) Prove the assertion of (ii) for centered maximal functions associated with general measures (with a possibly different constant).

3. On $(\mathbb{R}, \mu)$, consider the maximal function M_μ of Exercise 2.

(i) Prove the following covering lemma. Given a finite set $\mathcal{F}$ of open intervals in $\mathbb{R}$, prove that there exist two subfamilies $\mathcal{F}_1$, $\mathcal{F}_2 \subseteq \mathcal{F}$ each consisting of pairwise disjoint intervals such that the union of the intervals of $\mathcal{F}$ is equal to the union of the intervals of $\mathcal{F}_1$ and $\mathcal{F}_2$.

(ii) Use (i) to show the estimate

$$\mu(\{x \in \mathbb{R}^n : M_\mu f(x) > 1\}) \leq 2 \int_{\mathbb{R}} |f(x)| \mathrm{d}\mu(x).$$

Prove that the constant 2 cannot be improved.

4. Suppose that $f : [0,1] \times [0,1] \to (0,\infty)$ is a given function. Prove that there is a finite collection $\{A_j\}_{j \in J}$ of pairwise disjoint sets and a sequence $(x_j)_{j \in J}$ such that $\bigcup_{j \in J} A_j = [0,1]^2$, $x_j \in A_j$ and $\mathrm{diam}(A_j) \leq f(x_j)$.

5. Prove that there is an absolute constant C with the following property. If $F : [a,b] \to \mathbb{R}$ is a non-decreasing function, then for any $\lambda > 0$,

$$\left| \left\{ x \in [a,b] : \limsup_{h \to 0^+} \frac{F(x+h) - F(x)}{h} > \lambda \right\} \right| \leq C \cdot \frac{F(b) - F(a)}{\lambda}.$$

Remark: The function F may be discontinuous.

6. Let ν is a Borel measure, E be a subset of $\mathbb{R}$ and let B be a collection of intervals that cover E and satisfies $\sup_{I \in B} \nu(I) < \infty$. Prove that there is a disjoint sequence I_1, I_2, ... of intervals in B such that $\nu(E) \leq 5 \sum_j \nu(I_j)$.