Exercises in Measure Theory - 7

1. (i) A gambler plays the following game ten times: he flips a coin and if the outcome is tails, he wins 1; otherwise he loses 1. Prove that the total amount won by the gambler defines a signed measure on 2^Ω, where

$$\Omega = \{(\omega_1, \omega_2, \dots, \omega_{10}) : \omega_i \in \{-1, 1\}\}$$

is the associated sample space. Determine the Hahn and Jordan decomposition of this signed measure.

2. Suppose that $(X, \mathcal{M}, \mu)$ is a positive measure space and define a signed measure λ by

$$\lambda(A) = \int_A f \mathrm{d}\mu, \qquad A \in \mathcal{M},$$

where $f : X \to \mathbb{R}$ is a given summable function. Prove that for any $A \in \mathcal{M}$,

$$|\lambda|(A) = \int_A |f| \mathrm{d}\mu.$$

3. Suppose that μ, ν are finite measures on a σ-algebra $\mathcal{M}$ and let $E \in \mathcal{M}$ be a fixed set. Prove that for any $t \in \mathbb{R}$ there is an $A_t \in \mathcal{M}$ contained in E satisfying the conditions
(i) $\mu(F) \leq t\nu(F)$ whenever $F \in \mathcal{M}$ and $F \subset A_t$,
(ii) $\mu(F) \geq t\nu(F)$ whenever $F \in \mathcal{M}$ and $F \subset E \setminus A_t$.

4. Assume that μ is a signed measure on a σ-algebra $\mathcal{M}$ and let E be a measurable set. Prove that $\mu_+(E) = \sup\{\mu(F) : E \supseteq F \in \mathcal{M}\}$ and $\mu_-(E) = -\inf\{\mu(F) : E \supseteq F \in \mathcal{M}\}$.

5. Prove that the class of all finite Borel measures on $\mathbb{R}^n$, equipped with the norm defined by $||\mu|| = |\mu|(\mathbb{R}^n)$, forms a Banach space.

6. Suppose that λ, μ and ν are real measures defined on the same σ-algebra, satisfying the inequalities $\lambda \leq \mu$ and $\lambda \leq \nu$. Prove that $\lambda \leq \min(\mu, \nu)$, where $\min(\mu, \nu) = (\mu + \nu - |\mu - \nu|)/2$.

7. Let μ be a signed measure, and let $\mu = \mu_+ - \mu_-$ be the Hahn-Jordan decomposition. Prove that the following conditions are equivalent: (i) μ is finite, (ii) both measures μ_+, μ_- are finite.

8. Assume that $\mu : \mathcal{M} \to \mathbb{C}$ is a complex measure and f, $g : X \to \mathbb{R}$ are measurable functions. Show that $|\mu(f \in A) - \mu(g \in A)| \leq |\mu|(f \neq g)$ for any measurable $A \subseteq \mathbb{R}$.

9. Suppose that $f \in L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ and $g \in L^1(\mathbb{R}^n)$. Prove that $||f \star g||_{L^p} \leq ||g||_{L^1} ||f||_{L^p}$, where $f \star g$ denotes the convolution of f and g.

10. Let $\mu = \mu_+ - \mu_-$ be the Jordan decomposition of a signed measure μ and let ν, λ be two positive measures such that $\mu = \nu - \lambda$. Show that $\nu \geq \mu_+$ and $\lambda \geq \mu_-$.