Exercises in Measure Theory - 6

1. Let $f : X \to \mathbb{R}$ be a μ-measurable function such that $|f|^p$ is μ-summable for some $p > 0$. Prove that the sequence $(f\chi_{\{f<n\}})_{n\geq 1}$ converges in measure to f. What if we drop the assumption on the summability?

2. Suppose that (X, μ) is a measure space and $(f_n)_{n\geq 1}$, $(g_n)_{n\geq 1}$ are sequences of real-valued, measurable functions such that $f_n \to f$, $g_n \to g$ in measure.
(i) Is it true that $f_n + g_n \to f + g$ in measure?
(ii) Is it true that $f_n g_n \to fg$ in measure?
(iii) Is it true that $f_n \times g_n \to f \times g$ in measure (with respect to $\mu \times \mu$)?

3. Suppose that X is a countable set and μ is a measure on X. Prove that if $f_n \to f$ in measure, then $f_n \to f$ μ-almost everywhere. Is the reverse implication true?

4. Let (X, μ) be a measure space and, for $1 \leq p < \infty$, put

$$L^p(X, \mu) = \left\{ f : X \to \mathbb{R} : ||f||_p := \left(\int_X |f|^p \mathrm{d}\mu \right)^{1/p} < \infty \right\}.$$

Prove that $L^p(X, \mu)$, equipped with the metric $d(f, g) = ||f - g||_p$, is complete.

5. Suppose that (X, μ) is a measure space, g is a μ-summable function, a sequence $(f_n)_{n\geq 1}$ of measurable functions converges in measure to f and satisfies $|f_n| \leq g$ for all n. Prove that

$$\int_X |f_n - f| \mathrm{d}\mu \to 0.$$

6. Suppose that $(a_n)_{n\geq 1}$ is a sequence of real numbers, converging to $a \in \mathbb{R}$. Prove that the sequence $f_n(x) = \cos(a_n/x)$, $x \in \mathbb{R} \setminus \{0\}$, converges almost uniformly to $f(x) = \cos(a/x)$ (with respect to Lebesgue measure).

7. Show that almost uniform convergence implies convergence in measure.

8. Assume that $(f_n)_{n\geq 1}$ is a sequence of μ-measurable functions on X satisfying the condition $\sum_{n=1}^{\infty} \int_X |f_n| \mathrm{d}\mu < \infty$. Prove that $f_n \to 0$ almost uniformly.

9. Suppose that X is a countable set, μ is a measure on X and $(f_n)_{n\geq 1}$ is a sequence of measurable functions which converges to f in measure. Does $(f_n)_{n\geq 1}$ converge almost uniformly to f?