Exercises in Measure Theory - 5

1. Define $h : (0,1) \times (0,1) \to \mathbb{R}$ by

$$h(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}.$$

Show that

$$\int_0^1 \left(\int_0^1 h(x,y)\mathrm{d}y \right) \mathrm{d}x = \frac{\pi}{4}$$
$$\int_0^1 \left(\int_0^1 h(x,y)\mathrm{d}x \right) \mathrm{d}y = -\frac{\pi}{4}$$

and

$$\int_0^1 \int_0^1 |h(x,y)| \mathrm{d}x\, \mathrm{d}y = \infty.$$

2. Suppose that $f : [0,1] \to \mathbb{R}$ is a Lebesgue measurable function satisfying

$$\int_{[0,1]^2} |f(x) - f(y)|\, \mathrm{d}x\mathrm{d}y < \infty.$$

Prove that

$$\int_0^1 |f(x)|\mathrm{d}x < \infty.$$

3. A real-valued function $f(x,y)$, $x, y \in \mathbb{R}$, is a Borel function of x for every fixed y and a continuous function of y for every fixed x. Prove that f is a Borel function. Is the same conclusion true if we only assume that f is a real-valued Borel function in each variable separately?

4. Let $f : (0,\infty) \to \mathbb{R}$ be given by

$$f(t) = \int_0^\infty \frac{xe^{-x^2}}{x^2+t^2}\mathrm{d}x.$$

Compute $\lim_{t\to 0+} f(t)$ and $\int_0^\infty f(t)\mathrm{d}t$. Prove that f is differentiable.

5. Suppose that

$$f(t) = \int_0^\infty e^{-tx}\frac{\ln(1+x)}{1+x}\mathrm{d}x \qquad \text{for } t > 0.$$

(i) Prove that $\int_0^\infty f(t)\mathrm{d}t < \infty$.
(ii) Show that f is infinitely many times differentiable.

6. Suppose that $f : (0,1) \to \mathbb{R}$ is Lebesgue integrable.
(i) Prove that the function $g : (0,1) \to \mathbb{R}$, given by $g(x) = \int_x^1 t^{-1} f(t)\mathrm{d}t$ is continuous.
(ii) Show that $\int_0^1 g(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x$.

7. Let X be a space with measure μ and assume that $f : X \to \mathbb{R}$ is μ-measurable. Prove that the graph of f, i.e., the set $G = \{(x,y) \in X \times \mathbb{R} : y = f(x)\}$, is measurable with respect to the product measure $\mu \times |\cdot|$ and has measure zero.

8. Let X be a space with a σ-finite measure μ and assume that $\Phi : [0,\infty) \to [0,\infty)$ be a C^1 convex function such that $\Phi(0) = 0$. Prove that for any measurable function $f : X \to [0,\infty]$,

$$\int_X \Phi(f)\mathrm{d}\mu = \int_0^\infty \Phi'(t)\mu(\{x \in X : f(x) > t\})\mathrm{d}t = \int_0^\infty \Phi'(t)\mu(\{x \in X : f(x) \geq t\})\mathrm{d}t.$$