Exercises in Measure Theory - 3+4

1. (i) Assume that $f : X \to Y$ is μ-measurable. Show that $f^{-1}(B)$ is μ-measurable for any Borel set $B \subseteq Y$. Conclude that if $g : Y \to Z$ is Borel, then the composition $g \circ f$ is μ-measurable.
(ii) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable. Is $g \circ f$ Lebesgue measurable?

2. Let $f : X \to [0, \infty]$ be a μ-measurable function. Show that there is a nondecreasing sequence $(f_n)_{n \geq 1}$ of simple functions which converges to f μ-almost everywhere.

3. Let $f : X \to \mathbb{R}$ be a bounded μ-measurable function. Prove that there is a sequence $(f_n)_{n \geq 1}$ of simple functions which converges uniformly to f.

4. Assume that $f : X \to [0, \infty]$ is μ-measurable. Prove that there exist μ-measurable sets A_1, A_2, $\ldots \subseteq X$ such that

$$f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}.$$

5. Suppose that $\mu(X) < \infty$ and $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ are measurable functions. Show that there is a sequence $(\alpha_n)_{n \geq 1}$ of positive real numbers such that $\lim_{n\to\infty} \alpha_n f_n = 0$ $\mu-$almost everywhere.

6. Prove that $\int_0^{\infty} \frac{1}{x} \mathrm{d}x = +\infty$.

7. Let f, $(f_n)_{n \geq 1}$ be μ-measurable and let g, $(g_n)_{n \geq 1}$ be μ-integrable functions on X such that $f_n \to f$ μ-almost everywhere, $g_n \to g$ μ-almost everywhere, $|f_n| \leq g_n$ for each n and

$$\lim_{n\to\infty} \int_X g_n \mathrm{d}\mu = \int_X g \mathrm{d}\mu.$$

Prove that

$$\lim_{n\to\infty} \int_X |f_n - f| \mathrm{d}\mu = 0.$$

8. Let μ be a probability measure on X and let $f : X \to \mathbb{R}$ be a μ-measurable function.
(i) Prove that

$$\lim_{p\to\infty} \left(\int_X |f|^p \mathrm{d}\mu \right)^{1/p} = \operatorname{ess\,sup}_X |f|.$$

(ii) Prove that if f is μ-integrable, then

$$\lim_{p\to 0+} \left(\int_X |f|^p \mathrm{d}\mu \right)^{1/p} = \exp\left(\int_X \log |f| \mathrm{d}\mu \right).$$

9. Let $f_n : X \to [0, \infty]$, $n = 1, 2, \ldots$ be μ-measurable functions. Show that

$$\int_X \sum_{n=1}^{\infty} f_n \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X f_n \mathrm{d}\mu.$$

10. Suppose that $f : X \to \mathbb{R}$ is μ-integrable. Prove that $\lim_{t\to\infty} t\mu\big(\{x \in X : |f(x)| \geq t\}\big) = 0$.