Exercises in Measure Theory - 2

1. Prove that Lebesgue measure on $\mathbb{R}$ is Borel.

2. Suppose that μ^* is an outer measure on the class of all subsets of $\mathbb{R}$ such that every open set is μ^*-measurable. Show that for any A, $B \subseteq \mathbb{R}$ with $\operatorname{dist}(A, B) > 0$ we have

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

3. Let μ be a Radon measure on $\mathbb{R}^n$ and suppose that E is a Borel subset of $\mathbb{R}^n$. Prove that there are two Borel sets A, B such that $A \subseteq E \subseteq B$, A is F_σ, B is G_δ and $\mu(B \setminus A) = 0$.

4. Suppose that μ is a Radon measure on $\mathbb{R}$ and let $E \subset \mathbb{R}$ be a Borel set satisfying $\mu(E) > 0$. Prove that for any $0 \leq \alpha < 1$ there is an open interval $U \subset \mathbb{R}$ such that $\mu(E \cap U) \geq \alpha\mu(U)$.

5. Let $E \subset \mathbb{R}$ be a Borel set of positive measure. Show that there exists an open interval U containing the origin and entirely contained in the set $E - E = \{x - y : x,\, y \in E\}$.

6. Suppose that μ is a finite, nonnegative and finitely additive set function defined on an algebra $\mathcal{A}$ of subsets of $\mathbb{R}$. For any $A \subset \mathbb{R}$, put

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) \ : \ A \subseteq \bigcup_{n=1}^{\infty} E_n,\ E_n \in \mathcal{A}\right\}.$$

(a) Prove that μ^* is an outer measure.
(b) Is it true that $\mu(A) = \mu^*(A)$ for all $A \in \mathcal{A}$?
(c) What would be the answer in (b) if μ was assumed to be countably additive?

7. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing right-continuous function. Prove that there is a Radon measure μ on $\mathbb{R}$ such that $\mu((a, b]) = g(b) - g(a)$ for all $a < b$.

8. Let g be a Cantor function on $\mathbb{R}$ (i.e., a continuous function $g : \mathbb{R} \to [0, 1]$, which vanishes on $(-\infty, 0]$, is equal to 1 on $[1, \infty)$ and increases on the Cantor set $\mathcal{C}$). Let μ be the measure corresponding to g in the sense of Problem 7.
(a) Prove that for any $x \in \mathbb{R}$ we have $\mu(\{x\}) = 0$.
(b) Show that $\mu(\mathbb{R} \setminus \mathcal{C}) = 0$.

9. Let μ be the Lebesgue measure on $\mathbb{R}$ and let B be a Borel subset of $\mathbb{R}$ such that

$$\mu(E \setminus (E + x)) = 0 \qquad \text{for any } x \in \mathbb{R},$$

where $E + x = \{z + x : z \in E\}$. Prove that $\mu(E) = 0$ or $\mu(\mathbb{R} \setminus E) = 0$.