Exercises in Measure Theory - 13

1. Derive the Hausdorff measures of the following sets:
(i) the curve $\left\{\left(x, \sin\frac{1}{x}\right) : 0 < x < 1\right\} \subset \mathbb{R}^2$,
(ii) the curve $\left\{\left(x, \sin\frac{1}{x}\right) : 1/2 < x < 1\right\} \subset \mathbb{R}^2$,
(iii) the unit sphere in $\mathbb{R}^3$.

2. Derive the Hausdorff measure and the Hausdorff dimension of the Cantor set.

3. Prove the following properties of the Hausdorff dimension.
(i) if $X \subseteq Y$, then $\dim_H(X) \leq \dim_H(Y)$.
(ii) if X_i is a countable collection of sets with $\dim_H(X_i) \leq d$, then $\dim_H\left(\bigcup_i X_i\right) \leq d$.
(iii) if X is countable, then $\dim_H(X) = 0$.
(iv) if $X \subseteq \mathbb{R}^d$, then $\dim_H(X) \leq d$.
(v) if $f : X \to f(X)$ is a Lipschitz map, then $\dim_H(f(X)) \leq \dim_H(X)$.
(vi) if $\dim_H(X) = d$ and $\dim_H(Y) = d'$, then $\dim_H(X \times Y) \geq d + d'$.
(vii) if X is connected and contains more than one point, then $\dim_H(X) \geq 1$.

4. Compute the Hausdorff dimension of the following sets:
(i) A generalized Cantor set,
(ii) product of two Cantor sets,
(iii) the snowflake (Koch curve).

5. Suppose that E is a set with $\dim_H(E) = 0$. Does this imply that E is countable?