Exercises in Measure Theory - 12

1. Let $C(\mathbb{R})$ denote the class of all continuous, real-valued functions on $\mathbb{R}$. Suppose that $L : C(\mathbb{R}) \to \mathbb{R}$ is a linear functional satisfying $L(fg) = L(f)L(g)$ for any f, $g \in C(\mathbb{R})$. Show that $L(f) = f(x_0)$ for some $x_0 \in \mathbb{R}$.

2. Let μ_n, $n = 1, 2, \dots$ be Radon measures on $(X, \mathcal{B}(X))$. Determine whether μ_n is weakly convergent:
(i) $X = \mathbb{R}$, $\mu_n = \delta_n$.
(ii) $X = \mathbb{R}^d$, $\mu_n =$ Lebesgue measure on $[-n, n]^d$.
(iii) $X = \mathbb{R}^d$, $\mu_n = \frac{1}{n} \times$ Lebesgue measure on $[-n, n]^d$.
(iv) $X = \mathbb{R}^d$, $\mathrm{d}\mu_n = g_n \mathrm{d}x$, where $(g_n)_{n\geq 0}$ is a sequence of nonnegative functions which converges to g a.e. with respect to Lebesgue measure.

3. Let μ, μ_n, $n = 1, 2, \dots$ be Radon measures on $\mathbb{R}^d$ such that $\mu_n \rightharpoonup \mu$ and $\mu_n(\mathbb{R}^d) \to \mu(\mathbb{R}^d)$. Show that for any bounded continuous function f on $\mathbb{R}^d$ we have $\int_{\mathbb{R}^d} f \mathrm{d}\mu_n \to \int_{\mathbb{R}^d} f \mathrm{d}\mu$.

4. Let μ, μ_n, $n = 1, 2, \dots$, be Radon measures on $\mathbb{R}^d$. Prove that the following conditions are equivalent.
(i) $\mu_n \rightharpoonup \mu$.
(ii) $\int_{\mathbb{R}^d} f \mathrm{d}\mu_n \to \int_{\mathbb{R}^d} f \mathrm{d}\mu$ for every Borel bounded function f vanishing outside a certain ball, such that $\mu(\Delta_f) = 0$, where Δ_f is the set of all points where f is discontinuous.

5. Prove that the sequence $(\chi_{[n,n+1]})_{n\geq 1}$ converges weakly in $L^p(\mathbb{R})$, $1 < p < \infty$, but does not converge weakly in $L^1(\mathbb{R})$.

6. Assume that $(f_n)_{n\geq 0}$ is a sequence of functions in $L^p(\mathbb{R}^d)$, $1 < p < \infty$, which converges in measure to f. Does $(f_n)_{n\geq 1}$ converge weakly in $L^p(\mathbb{R}^d)$ to f?