Exercises in Measure Theory - 11

1. Let μ be a complex measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.

(i) Show that μ can be expressed as $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where μ_1, μ_2, μ_3 and μ_4 are finite measures.

(ii) Show that there is a measure ν and a complex-valued ν-measurable function φ with $|\varphi| = 1$ such that for any Borel set E,

$$\mu(E) = \int_E \varphi \mathrm{d}\nu.$$

(iii) Prove that the measure ν in (ii) is unique and that φ is uniquely determined up to a set of ν-measure 0.

(iv) Prove that if μ satisfies $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n) = 1$, then μ is a positive measure.

2. Let μ be a σ-finite positive measure on (X, M) and let $(f_n)_{n\geq 1}$ be a sequence of measurable functions which converges in μ-measure to a measurable function f. Moreover, suppose that ν is a finite positive measure on (X, M) such that $\nu << \mu$. Prove that $f_n \to f$ in ν-measure.

3. Prove that if $A \subset I = [0, 1]^n$ satisfies $\mu(A) < 1$ (where μ is the Lebesgue measure), then for any $\varepsilon > 0$ there is a cube $Q \subset I$ such that $\mu(A \cap Q) < \varepsilon\mu(Q)$.

4. For each $h > 0$, let E_h be a subset of $B(0, h)$ with the property that $\mu(E_h) \geq c\mu(B(0, h))$ for some $c > 0$ independent of h (μ denotes the Lebesgue measure). Show that if $f : \mathbb{R}^d \to \mathbb{C}$ is locally integrable, and x is a Lebesgue point of f, then

$$\lim_{h\to 0} \frac{1}{\mu(E_h)} \int_{x+E_h} f(y)\mathrm{d}y = f(x).$$

5. Let U be an open set in $\mathbb{R}^2$.

(i) Is it true that the set of Lebesgue density points equals $\operatorname{int} U$?

(ii) Is it true that the set of Lebesgue density points equals $\operatorname{int}(\operatorname{cl} U)$?

6. Suppose that $\mathcal{C}$ is the following fat Cantor set: from $[0, 1]$, remove the interval $(3/8, 5/8)$ of length $1/4$; then, remove the ,,center" intervals of length $1/16$ from $[0, 3/8]$ and $[5/8, 1]$, i.e., $(5/32, 7/32)$ and $(25/32, 27/32)$. Next, from each of the four connected parts, remove four ,,center" intervals of length $1/64$ each; continue this procedure.

Is it true that each point which is not an endpoint of an interval thrown out during the above construction, is the Lebesgue density point of $\mathcal{C}$?