Exercises in Measure Theory - 10

1. Compute the following derivatives $D_\mu \nu$:
(i) Let ν be the Dirac measure in $0 \in \mathbb{R}$, μ is the Lebesgue measure.
(ii) Let ν be the Lebesgue measure on $\{(x,y) : y = x\}$, μ is the Lebesgue measure on $\mathbb{R}^2$.

2. Let μ, ν be Radon measures on $\mathbb{R}^n$ and let α be a positive number. Prove that

$$A \subset \{x \in \mathbb{R}^n : \overline{D}_\mu \nu(x) \geq \alpha\}$$

implies $\nu(A) \geq \alpha\mu(A)$.

3. Let μ be the Lebesgue measure on $\mathbb{R}$ and let ν be the Lebegue-Stieltjes measure associated with the Cantor function $g : [0,1] \to [0,1]$. Compute $D_\mu \nu$.

4. Let $f : [0,1) \to \mathbb{R}$ be a Lebesgue summable function. Let $\mathcal{D}$ be the collection of dyadic intervals, i.e., $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}^n$, where

$$\mathcal{D}^n = \{[0, 2^{-n}), [2^{-n}, 2 \cdot 2^{-n}), \ldots, [(2^n - 1) \cdot 2^{-n}, 1)\}.$$

Using martingale theory, prove that for μ almost all x,

$$\lim_{n \to \infty} \frac{1}{\mu(D_n)} \int_{D_n} f(y) \mathrm{d}y = f(x).$$

5. Let μ, ν be signed measures such that ν is absolutely continuous with respect to μ. Does it necessarily imply that $\mu(E) = 0 \Rightarrow \nu(E) = 0$?