Exercises in Measure Theory - 1

1. Are the functions μ^* outer measures on 2^X? If so, identify the class of measurable subsets with respect to μ^* and check the regularity of μ^*.

a) X is an arbitrary set, $\mu^*(E) = 1$ for all E.

b) X is an arbitrary set, $x_0 \in X$ is a fixed point, $\mu^*(E) = \chi_E(x_0)$ for $E \in 2^X$.

c) $X = \{0, 1\}$, $\mu^*(\emptyset) = 0$, $\mu^*(\{0\}) = 1$, $\mu^*(\{1\}) = 2$, $\mu^*(\{0, 1\}) = 2$.

d) X is the set of 100 points arranged in an array 10×10; for all $E \in 2^X$, $\mu^*(E)$ is a number of columns which intersect E.

e) $X = \mathbb{N}$; for $E \in 2^X$, we take

$$\mu^*(E) = \limsup_{n\to\infty} \frac{\#\{E \cap \{1, 2, \ldots, n\}\}}{n}.$$

f) X is an arbitrary set, $\mu^*(E) = \#E$ for all E.

g) X is an arbitrary set, for $E \in 2^X$ put

$$\mu^*(E) = \begin{cases} 0 & \text{jeśli } E \text{ jest co najwyżej przeliczalny,} \\ +\infty & \text{jeśli } E \text{ jest nieprzeliczalny.} \end{cases}$$

2. Suppose that X is an arbitrary set, μ_1^* is a regular outer measure on 2^X and μ_2^* is given by

$$\mu_2^*(E) = \begin{cases} 0 & \text{jeśli } E = \emptyset, \\ 1 & \text{jeśli } E \neq \emptyset, \end{cases}$$

for $E \subseteq X$. Prove that $\mu_1^* + \mu_2^*$ is an outer measure which is not regular in general.

3. Suppose that X is an arbitrary set, μ^* is an outer measure on 2^X and take $A \subseteq X$. Prove that any measurable set B with respect to μ^* is measurable with respect to the restriction $\mu^* \llcorner A$.

4. Suppose that μ^* is an outer measure on 2^X and pick $A \subseteq C \subseteq X$. Prove that if B is μ^*-measurable and satisfies $A \subseteq B$ and $\mu^*(A) = \mu^*(B)$, then $\mu^*(C) = \mu^*(B \cup C)$.

5. Prove that there is no outer measure μ^* on $\mathbb{Q}$ such that

$$\mu^*(\{x \in \mathbb{Q} : a \le x \le b\}) = b - a$$

dla wszystkich a, $b \in \mathbb{Q}$, $a < b$.

6. Let μ be a measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and a measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that for all s, $t \in \mathbb{R}$ we have

$$\mu((-\infty, s] \times [t, \infty)) = \nu((-\infty, s])\nu([t, \infty)).$$

Prove that for all A, $B \in \mathcal{B}(\mathbb{R})$ we have the identity $\mu(A \times B) = \nu(A)\nu(B)$.

7. Prove that if a monotone class $\mathcal{M}$ contains an algebra $\mathcal{K}$, then it contains the σ-algebra $\sigma(\mathcal{K})$ as well.

8. Suppose that μ is a Borel measure on $[0, 1]$ and $|\cdot|$ is the Lebesgue's measure on $[0, 1]$. Show that if for any Borel set $A \subset [0, 1]$ satisfying $|A| = 1/2$ we have $\mu(A) = 1/2$, then $\mu = |\cdot|$.

9. For $k \ge 1$ define $A_k = \{0, k, 2k, \ldots\}$. Prove that there is no measure μ on $\mathbb{N}$ satisfying $\mu(A_k) = 1/k$ for all $k \ge 1$.

10. We say that a set $E \subseteq \mathbb{R}$ has an infinite density point, if there is an uncountable number of its elements outside any bounded interval. Define $\mu^* : 2^{\mathbb{R}} \to [0, \infty]$ by

$$\mu^*(E) = \begin{cases} 0 & \text{if } E \text{ is at most countable,} \\ 1 & \text{if } E \text{ is uncountable, but does not have an infinite density point,} \\ \infty & \text{if } E \text{ has an infinite density point.} \end{cases}$$

Prove that μ^* is an outer measure and identify the class of measurable sets. Is μ^* regular?