Bellman function method in probability and analysis - Joint Homework

Deadline: September 4th, 2017.

1. Let $n \geq 1$ be a fixed integer. Using the dynamic programming approach, find

$$\inf \left\{ a_1^2 + a_2^4 + a_3^8 + \ldots + a_n^{2^n} \right\},$$

where the infimum is taken over all positive numbers a_1, a_2, $\ldots$, a_n satisfying $a_1 a_2 \ldots a_n = 1$.

2. Let f be a martingale and g be its transform by a predictable sequence with values in $[-1, 1]$. Prove that if $|g_\infty| \geq 1$ almost surely (where g_∞ is the almost sure limit of g), then $\sup_n ||f_n||_4 \geq C$ for some absolute positive constant C. Find the best (i.e., the largest) C.

3. Let f be a nonnegative martingale and let f^* denote its maximal function. Find the best constant C in the estimate

$$\mathbb{P}(f_n + f_n^* \geq 1) \leq C \mathbb{E} f_n^2, \qquad n = 0, 1, 2, \ldots.$$

4. Let X, Y be continuous-time orthogonal martingales such that Y is differentially subordinate to X and $||X||_\infty \leq 1$. Find the best constant C in the inequality

$$\mathbb{E} Y_t^4 \leq C, \qquad t \geq 0.$$