Bellman function method in probability and analysis - Homework 2

Deadline: June 23th, 2017.

1. For $0 < p < 1$, find the best constant in the martingale inequality

$$||Y_t||_p \leq C_p ||X_t||_1, \qquad t \geq 0,$$

if X, Y are assumed to be orthogonal, X is nonnegative and Y is differentially subordinate to X.

2. For any $K > 0$, find the best constant $L(K)$ such that the following holds. If $f = (f_n)_{n\geq 0}$ is a simple nonnegative martingale, then

$$||f_n^*||_1 \leq K\mathbb{E}(f_n + 1)\log(f_n + 1) + L(K), \qquad n = 0,\, 1,\, 2,\, \ldots.$$

3. Let $\mathcal{H}^{\mathbb{T}}$ denote the periodic Hilbert transform and let $f : (-\pi, \pi] \to \mathbb{R}$ be a measurable function satisfying $||f||_{L^\infty} \leq 1$. For a given $\lambda > 0$, find the optimal constant C_λ depending only on λ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(\lambda \mathcal{H}^{\mathbb{T}} f(x)) \mathrm{d}x \leq C_\lambda.$$

4. Let $\mathcal{D}$ denote the collection of all dyadic subintervals of the unit interval $[0, 1]$. Find the best absolute constant C such that the following holds. If $(\alpha_Q)_{Q\in\mathcal{D}}$ is a sequence of nonnegative numbers satisfying

$$\sum_{Q\subseteq R,\, Q\in\mathcal{D}} \alpha_Q \leq |R| \qquad \text{for any } R \in \mathcal{D},$$

then for any integrable function $f : [0, 1] \to \mathbb{R}$ we have

$$\sum_{Q\in\mathcal{D}} \alpha_Q \exp\left(\frac{1}{|Q|}\int_Q f(x)\mathrm{d}x\right) \leq C \int_0^1 e^{f(x)} \mathrm{d}x.$$