Bellman function method in probability and analysis - Homework

Deadline: May 10th, 2017.

1. Using the dynamic programming, prove that for any positive numbers $a_1, a_2, \ldots, a_n$ satisfying $a_1 a_2 \ldots a_n = 1$ we have

$$\frac{1}{n-1+a_1} + \frac{1}{n-1+a_2} + \ldots + \frac{1}{n-1+a_n} \leq 1.$$

2. Let $1 < p < \infty$ and $\alpha < p-1$. Using the dynamic programming, find the best constant $C_{p,\alpha}$ in the estimate

$$\int_0^\infty \left| \frac{1}{t} \int_0^t f(u)\mathrm{d}u \right|^p t^\alpha \mathrm{d}t \leq C_{p,\alpha} \int_0^\infty |f(t)|^p t^\alpha \mathrm{d}t,$$

to be valid for all $f \in L^p(0,\infty)$.

3. Let $(S_n)_{n\geq 0}$ be a symmetric random walk over the integers and let $\beta \in (0,1)$ be a fixed parameter. Solve the optimal stopping problem

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_x \left[\beta^\tau (1 - \exp(S_\tau))^+ \right], \qquad x \in \mathbb{Z}$$

(here, as usual, $\mathbb{P}_x$ is the probability under which S_0 starts from x, and $\mathbb{E}_x$ is the associated expectation).

4. Let $n \geq 1$ be a fixed integer. Unknown, pairwise distinct numbers $a_1, a_2, \ldots, a_n$ were written on n balls in an urn. The balls were drawn without replacement from an urn, one by one. At each time we may stop and take the number which is written on the ball we have just obtained, or we can discard the ball and continue (if the urn is nonempty). Our goal is to maximize the probability that the number we take equals the maximum of $a_1, a_2, \ldots, a_n$. Find the stopping procedure which solves this problem.

5. For any $\lambda > 0$, find the best constant $C_\lambda \in (0, \infty]$ such that if f is a martingale taking values in $[-1,1]$ and g is differentially subordinate to f, then

$$\mathbb{E} \exp(\lambda g_n) \leq C_\lambda, \qquad n = 0, 1, 2, \ldots.$$