THE UNIVERSITY OF CHICAGO

HOMOTOPICAL LOCALIZATIONS AT A SPACE

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BY
ADAM J. PRZEŹDZIECKI

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ABSTRACT

By localization we understand a coaugmented idempotent functor. For a sufficiently good space Z, for example a profinitely completed one, we construct a localization at Z which sends to weak equivalences precisely those maps f for which $\mathrm{map}_*(f, Z)$ is a weak equivalence. Such notion is dual to the Bousfield and Farjoun f-localization.

We apply this construction to show that there is a terminal localization through which the Sullivan profinite completion factors.

We also introduce relative phantom maps which allow us to compare some homotopy colimit constructions with homotopy limit constructions. We use them in our attempt to show that for the spaces Z as above, localization at Z can be constructed out of Z by means of homotopy limits.

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CHAPTER 1
INTRODUCTION

Our main motivation for the work presented in this dissertation is to construct a localization functor, in a certain sense dual to the f-localization of Bousfield and Farjoun, and to study some of its properties. We succeed in a case which is related to the Sullivan profinite completion.

We can view f-localization as the initial coaugmented idempotent functor on the homotopy category which takes a map f to an equivalence. In [2] Bousfield used the small object argument to prove that f-localizations exist for all maps f. The role of these functors was especially exposed in 1990's when they were put in a convenient framework in terms of mapping complexes. A survey of related methods can be found in [12] and [7]. It seems natural to ask if a dual notion of a localization at a space Z, that is the terminal idempotent functor with a given space Z in its image (Definition 2.5), might not also be interesting. The main reason these localizations have not been considered very much is that they are not known to exist in general, even in the stable case (see Chapter 7 in [14]).

As every homological localization can be realized as an f-localization, every cohomological localization, provided it exists, is a localization at a suitable space. Research towards establishing the existence of cohomological localizations was briefly summarized in 2.6 of [6].

Here we prove the existence of localizations at fibrant simplicial compact spaces, where compact means compact Hausdorff (Theorem 4.7). Examples of such spaces include the ones which are profinite completions of another space, mapping complexes with a profinitely completed target, and others. This result allows us to construct an idempotent approximation to the Sullivan profinite completion (Theorem 6.2). Possibly the most interesting technique developed here is the use of relative phantom maps (Definitions 7.1 and 7.6) to show that, in the cases we consider, it should be possible to construct localization at a space by means of iterated homotopy limits (Theorem 9.11) as well as by means of iterated homotopy colimits. Unfortunately we need certain technical assumption (Conjecture 8.3) to be able to form compositions of those phantom maps. We hope to be able to prove this assumption in the future.

We would like to be able to prove the existence of localization at an arbitrary space without relying on the compactness condition, and there is some evidence that such localizations should exist at abelian Eilenberg-Mac Lane spaces. These would form "truncated localizations at an ordinary cohomology theory", an analogue of "truncated localizations at a homology theory" whose existence was shown by Ohkawa in [17]. It would also be interesting to find how such localizations act on spaces and how they are related to those f-localizations, if any, that do not correspond to a localization at any space.

In 1998 Casacuberta, Scevenels and Smith showed (see [9]) that a more general result, from which the existence of localizations at any space would follow, actually depends on certain large cardinal axioms. More precisely their result is true under assumption of the Vopenka principle but false under assumption of nonexistence of the measurable cardinal numbers. Despite extensive efforts we were unable to avoid similar problems in our attempts to prove the existence of localizations at a general space, nor were we able to disprove it under some large cardinal axioms.

The exposition is organized as follows. Chapters $1-6$ are self contained and their aim is to prove Theorem 4.7 and derive some consequences. Chapters $7-9$ are aimed at the proof of Theorem 9.11. Chapter 7 is a self contained basic theory of relative phantom maps. Chapters 8 and 9 are written under assumption of Conjecture 8.3. This conjecture is quite technical and probably of little interest in itself but it is meant to serve us as an excuse to present methods which, I believe, are close to provide an actual proof of Theorem 9.11.

The dissertation is written simplicialy. We use terms space and simplicial set as synonyms choosing the second one wherever confusion with compact topological space might occur.

CHAPTER 2

LOCALIZATIONS

In this chapter we collect basic definitions and facts related to homotopy localizations.

A functor L is called *coaugmented* if it comes with a natural transformation $\eta_X : X \to LX$ from the identity to L. A coaugmented functor is *idempotent* if in the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & LX \arrow[d, "\eta_{LX}"] \\
LX \arrow[r, "L\eta_X"] & LLX
\end{tikzcd}$$

the maps η_{LX} and $L\eta_X$ are equivalences and $\eta_{LX} = L\eta_X$.

Definition 2.1. A coaugmented idempotent functor is called a *localization*.

Although this definition makes sense in any category we will consider only localizations in the homotopy category Ho_* of pointed simplicial sets. A pointed simplicial set Z is said to be *L-local* if the map $\eta_Z : Z \to LZ$ is an equivalence. It is straightforward to check that the class of L-local spaces uniquely determines and is determined by the functor L. A map $g : X \to Y$ is an *L-equivalence* if Lg is an equivalence. There is a natural ordering of localizations as described below.

Definition 2.2. Given two localization functors L_1 and L_2 we say that $L_1 \leq L_2$ if one of the equivalent conditions hold:

(i) there is a natural transformation $L_1 \to L_2$ giving $L_2L_1 \simeq L_2$

(ii) any L_1-equivalence is also an L_2-equivalence

(iii) any L_2-local space is also L_1-local

The work of Casacuberta, Scevenels and Smith [9] implies that under the large cardinal axiom called Vopenka principle this ordering is naturally isomorphic to the Bousfield lattice of f-localizations (4.3 in [6]).

Given a map $f : A \to B$ we say that a fibrant pointed simplicial set Z is *f-local* if the induced map of function complexes

$$f^* : \mathrm{map}_*(B, Z) \to \mathrm{map}_*(A, Z) \tag{2.3}$$

is an equivalence. If Z is connected the condition above is equivalent to the one that the induced map of unbased function complexes

$$f^* : \mathrm{map}(B, Z) \to \mathrm{map}(A, Z)$$

is an equivalence.

A map $g : X \to Y$ is an *f-equivalence* if any f-local space is also g-local. This means that for any fibrant simplicial set Z if

$$f^* : \mathrm{map}_*(B, Z) \xrightarrow{\simeq} \mathrm{map}_*(A, Z)$$

then

$$g^* : \mathrm{map}_*(Y, Z) \xrightarrow{\simeq} \mathrm{map}_*(X, Z).$$

Definition 2.4. An *f-localization* is a localization functor L_f such that the following conditions hold:

(i) The classes of f-equivalences and L_f-equivalences coincide.

(ii) The classes of f-local and L_f-local simplicial sets coincide.

(iii) The map $X \to L_f X$ is an f-equivalence and $L_f X$ is f-local.

(iv) L_f is the initial localization functor such that the map f is an L_f-equivalence.

For a map f, there are obvious implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$.

The existence of f-localizations for arbitrary maps f was proved by Bousfield [2] and Farjoun [11].

Let Z be a fibrant simplicial set. We say that a map $g : X \to Y$ is a *Z-equivalence* if the induced map of function complexes

$$g^* : \mathrm{map}_*(Y, Z) \to \mathrm{map}_*(X, Z)$$

is an equivalence. A fibrant simplicial set K is *Z-local* if it is g-local for all Z-equivalences g. This means that for any g if

$$g^* : \mathrm{map}_*(Y, Z) \overset{\simeq}{\to} \mathrm{map}_*(X, Z)$$

then

$$g^* : \mathrm{map}_*(Y, K) \overset{\simeq}{\to} \mathrm{map}_*(X, K)$$

Definition 2.5. A *localization at Z* is a localization functor L_Z such that the following conditions hold:

(i) the classes of Z-equivalences and L_Z-equivalences coincide.

(ii) the classes of Z-local and L_Z-local simplicial sets coincide.

(iii) The map $X \to L_Z X$ is a Z-equivalence and $L_Z X$ is Z-local.

(iv) L_Z is the terminal localization functor such that the space Z is L_Z-local.

For a space Z, there are obvious implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$.

The existence of localization at a given space Z is not known in general. It is clear that the classes of Z-equivalences and f-equivalences are closed under arbitrary homotopy colimits. Also the classes of Z-local and f-local spaces are closed under arbitrary homotopy limits.

Lemma 2.6. *Suppose that for a certain space Z there is a set of Z-equivalences $\{f_\alpha\}$ such that every Z-equivalence can be presented as a homotopy colimit of elements of the set $\{f_\alpha\}$. Then the localization at Z is simply an f-localization for $f = \bigvee f_\alpha$.*

The following lemma will "streamline" many arguments.

Lemma 2.7. *A cofibration $f : A \hookrightarrow B$ is a Z-equivalence if and only if every diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "f"'] & \prod_{n\geq 0} map_*(\Delta^n_+, Z) \arrow[d, two heads, "p"] \\
B \arrow[r] \arrow[ur, dashed] & \prod_{n\geq 0} map_*(\partial\Delta^n_+, Z)
\end{tikzcd}$$

admits a lift marked by the dashed arrow.

Proof. Since the map p is a product of maps $p_n : \mathrm{map}_*(\Delta^n_+, Z) \twoheadrightarrow \mathrm{map}_*(\partial\Delta^n_+, Z)$ for $n \geq 0$ the claim of the lemma is that a map $f : A \hookrightarrow B$ is a Z-equivalence if and only if f has the left lifting property with respect to all the maps p_n.

In general given a cofibration $g : C \hookrightarrow D$ we use adjointness to note that the existence of a dashed lift in the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "f"] & \mathrm{map}_*(D, Z) \arrow[d, two heads, "g^*"] \\
B \arrow[r] \arrow[ur, dashed] & \mathrm{map}_*(C, Z)
\end{tikzcd}$$

is equivalent to the existence of the dashed map in the following commutative diagram.

$$\begin{tikzcd}
A \wedge D \arrow[d, hook, "f\wedge id"'] \arrow[dr] & & A \wedge C \arrow[ll, hook', "id\wedge g"'] \arrow[d, hook, "f\wedge id"] \\
 & Z & \\
B \wedge D \arrow[ur, dashed] & & B \wedge C \arrow[ll, hook', "id\wedge g"] \arrow[ul]
\end{tikzcd}$$

This in turn is equivalent to the lifting property as indicated on the next diagram.

$$\begin{tikzcd}
C \arrow[r] \arrow[d, hook, "g"] & \mathrm{map}_*(B, Z) \arrow[d, two heads, "f^*"] \\
D \arrow[r] \arrow[ur, dashed] & \mathrm{map}_*(A, Z)
\end{tikzcd}$$

To prove the claim of the lemma we specify g to be any of the inclusion maps $g_n : \partial\Delta^n_+ \hookrightarrow \Delta^n_+$ for $n \geq 0$. We proved that the diagram in the lemma admits the lift as indicated if and only if f^* has the right lifting property with respect to all the maps g_n. Since f^* is a fibration, this in turn happens if and only if f^* is a weak equivalence. $\square$

CHAPTER 3

MAPS WITH A COMPACT TARGET

In this chapter we describe natural compact topologies that arise on the sets of simplicial maps, and on certain function complexes with a simplicial set (i.e. a simplicial discrete space) as a source and a simplicial compact space as a target. By compact we always mean compact Hausdorff. We end the chapter by describing natural compact topologies on homotopy classes of maps with similar source and target.

We work with pointed simplicial sets but we want to distinguish those which admit a levelwise compact topology.

Definition 3.1. We say that Z is a *simplicial compact space* if there is a simplicial object $\overline{Z}$ in the category of compact topological spaces such that the forgetful functor from topological spaces to sets takes $\overline{Z}$ to Z. When we use the term simplicial compact space we consider the choice of $\overline{Z}$ as part of the structure.

Definition 3.2. A map $g : S \to T$ is *compact* if the corresponding map $\overline{S} \to \overline{T}$ is continuous. A pair (Z, Z_0) is *compact* if the corresponding inclusion map $Z_0 \hookrightarrow Z$ is compact.

Note that saying that (Z, Z_0) is compact is stronger than saying that both Z and Z_0 are compact.

Until Lemma 3.8, that is as long as we don't work with homotopy classes of maps, we do not need to assume that Z is fibrant.

Lemma 3.3. *Let X be a simplicial set and Z a simplicial compact space. The set* $\hom_*(X, Z)$ *has a natural compact topology.*

Proof. To see this observe that $\hom_*(X,Z)$ is a subset of

$$\prod_n \mathrm{Sets}(X_n, Z_n) \cong \prod_n \prod_{X_n} Z_n$$

which has a compact product topology. The space $\hom_*(X,Z)$ is determined by a number of equations (see May [15] 1.2) between continuous maps so it forms a closed hence compact subspace of the product. To show naturality in both variables we note that the maps of $\hom_*$ sets induced by the maps of the variables are restrictions of the appropriate continuous maps of the products above. $\square$

Given two maps $f : A \to B$ and $g : S \to T$ let $\hom_*(f,g)$ denote the set of commutative diagrams

$$\begin{tikzcd} A \arrow[r, "h_A"] \arrow[d, "f"'] & S \arrow[d, "g"] \\ B \arrow[r, "h_B"] & T \end{tikzcd}$$

Lemma 3.4. *If $f : A \to B$ is any map and $g : S \to T$ is compact then the set $\hom_*(f,g)$ has a natural compact topology*

Proof. The set $\hom_*(f,g)$ is the equalizer of the diagram

$$\begin{tikzcd} \hom_*(A,S) \times \hom_*(B,T) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \hom_*(A,T) \end{tikzcd}$$

where $\alpha(h_A, h_B) = g \circ h_A$ and $\beta(h_A, h_B) = h_B \circ f$ are continuous maps. The claim follows from compactness of the source and target of maps α and β and from the naturality of this diagram. $\square$

Let (Z, Z_0) be a compact pair and (X, X_0) a pair of simplicial sets (i.e. simplicial discrete spaces).

Corollary 3.5. *The set $\hom_*((X,X_0),(Z,Z_0))$ of relative simplicial maps has a natural compact topology.*

Corollary 3.6. *For (X,X_0) and (Z,Z_0) as in 3.5 the function complex $map_*((X,X_0),(Z,Z_0))$ has a natural structure of a simplicial compact space.*

Proof. For the unbased function complex $\mathrm{map}((X,X_0),(Z,Z_0))$ it is enough to notice that the set of its n-dimensional simplices is given ([15] 6.4) by

$$\begin{aligned}&\mathrm{hom}((X\times\Delta^n,X_0\times\Delta^n),(Z,Z_0))=\\&\qquad\qquad=\mathrm{hom}_*(((X\times\Delta^n)_+,(X_0\times\Delta^n)_+),(Z,Z_0))\end{aligned}$$

which by 3.5 has a natural compact topology. The function complex in question is the inverse image of a base point under the continuous map between compact spaces below.

$$\mathrm{map}((X,X_0),(Z,Z_0))\to\mathrm{map}((*,*),(Z,Z_0))$$

□

We note in particular that $\Omega Z=\mathrm{map}_*(S^1,Z)$ is again a simplicial compact space.

Fix a map $f:(X,X_0)\to(Z,Z_0)$. Let $\mathrm{C}(X,X_0)$ be the set of homotopies relative to X_0 that contract f to a trivial map (one that factors as $(X,X_0)\to(Z_0,Z_0)\to(Z,Z_0)$).

Lemma 3.7. *If (X,X_0) is a simplicial pair and (Z,Z_0) is a compact pair then $C(X,X_0)$ has a natural compact topology.*

Proof. Let $I=\Delta^1$ be the simplicial set whose realization is a unit interval. Let c be the composite

$$(X\times\{0\}\cup X_0\wedge I_+,X_0\wedge I_+)\to(X,X_0)\xrightarrow{f}(Z,Z_0)$$

where the first map is the projection on X. We observe that $\mathrm{C}(X,X_0)=\gamma^{-1}(\{c\})$ where

$$\begin{aligned}&\gamma:\mathrm{hom}_*((X\wedge I_+,X\times\{1\}\cup X_0\wedge I_+),(Z,Z_0))\to\\&\qquad\qquad\to\mathrm{hom}_*((X\times\{0\}\cup X_0\wedge I_+,X_0\wedge I_+),(Z,Z_0))\end{aligned}$$

is a restriction map and thus a continuous map between compact spaces. □

In the rest of this chapter we show that the compact structures introduced above pass to the homotopy category. This will not be used in the proofs of our main theorems but it is an interesting property of fibrant simplicial compact spaces.

We will use a closure operation C on the family of subsets of Z_0, the 0-simplices of Z. We define C by setting $C(X) = \partial_1 \partial_0^{-1}(X)$ where ∂_i are the face maps from Z_1 to Z_0.

Lemma 3.8. *If Z is a fibrant pointed simplicial compact space then the closure operation C defined above has the following properties.*

(i) *if $X \subseteq Z_0$ is a closed subset then the set $C(X) \subseteq Z_0$ is also closed.*

(ii) *two vertices $x, y \in Z_0$ are in the same connected component of Z if and only if $x \in C(\{y\})$.*

(iii) $X \subseteq C(X)$

(iv) $C(C(X)) = C(X)$

(v) *if $Y \cap C(X) = \varnothing$ then $C(Y) \cap C(X) = \varnothing$.*

Proof. Property *(i)* is a consequence of the compactness of the spaces Z_i and the continuity of the maps ∂_i for $i = 0, 1$.

To show *(ii)* we observe that $C(X)$ is the set of those 0-simplices $y \in Z_0$ for which there is some 1-simplex $\sigma \in Z_1$ such that

$$\partial_1 \sigma = y \text{ and } \partial_0 \sigma \in X.$$

This is the same as to say that there is a path (i.e. 1-simplex) from an element in X to y.

Properties *(iii)* through *(v)* follow directly from the fact that $C(X)$ is the union of 0-simplices of the connected components of Z that intersect X. This description of $C(X)$ follows from the proof of *(ii)*.

The assumption that Z is fibrant was used implicitly. We needed to know that if there is a path from x to y then there is a path from y to x and if there are paths from x to y and from y to z then there is a path from x to z. □

Proposition 3.9. *If Z is a fibrant pointed simplicial compact space then the set $\pi_0 Z$ has a natural compact topology.*

Proof. The space $\pi_0 Z$ is a quotient of the space Z_0 of 0-simplices of Z. A subset of $\pi_0 Z$ is open if its inverse image in Z_0 is open. Since Z_0 is compact the property that every open cover of $\pi_0 Z$ contains a finite subcover is obvious. The non-obvious part is that $\pi_0 Z$ is Hausdorff.

All references in the remainder of this proof are to Lemma 3.8. Let $\pi : Z_0 \to \pi_0 Z$ be the projection from the 0-simplices to connected components of Z. By *(ii)* we have $C(X) = \pi^{-1}(\pi(X))$ for any subset X of Z_0. Consider two points $\overline{z}_0 \neq \overline{z}_1$ in $\pi_0 Z$. Their inverse images are of the form $C_0 = C(\{z_0\})$ and $C_1 = C(\{z_1\})$ for some z_0 and z_1 in Z_0. By *(i)* the sets C_0 and C_1 are closed in Z_0. The space Z_0 is compact hence normal. That is for two disjoint closed subsets C_0 and C_1 in Z_0, there are closed subsets K_0 and K_1 satisfying

$$C_0 \cap K_0 = \varnothing,\ C_1 \cap K_1 = \varnothing \text{ and } K_0 \cup K_1 = Z_0$$

By *(iv)* $C(C_i) = C_i$ for $i = 0, 1$ so by *(v)* we obtain $C(K_i) \cap C_i = \varnothing$ for $i = 0, 1$. Also by *(iii)* $K_i \subseteq C(K_i)$ hence $C(K_0) \cup C(K_1) = Z_0$. The sets $C(K_i)$ are closed by *(i)* hence we obtain two disjoint open sets $V_i = Z_0 \smallsetminus C(K_i)$ that separate C_0 and C_1. Finally by *(v)* and *(iii)* we get $C(V_i) = V_i$. We see that the images of V_i in $\pi_0 Z$ define open sets that separate $\overline{z}_0$ and $\overline{z}_1$. This proves that $\pi_0 Z$ is Hausdorff. ☐

Corollary 3.10. *If Z and Z_0 are fibrant simplicial compact spaces then the set $[(X, X_0), (Z, Z_0)]$ has a natural compact topology.*

Since any compact topological group is either finite or uncountable we get the following statement.

Corollary 3.11. *If Z is a fibrant simplicial compact space, X any simplicial set and $n > 0$ then $\pi_n map_*(X, Z)$ is either finite or uncountable.*

CHAPTER 4

LOCALIZATIONS AT A SPACE

In this chapter we will prove (Theorem 4.7) that localization at a space Z exists whenever Z is a fibrant retract of a simplicial compact space. We attain this by showing that for such spaces Z any Z-equivalence can be presented as a filtered colimit of Z-equivalences of bounded cardinalities so that we can use Lemma 2.6. The simplicial sets are not assumed to be fibrant.

Let $\mathcal{S}_*^2$ be the usual category of maps in $\mathcal{S}_*$ (pointed simplicial sets). We will say that $f_0 : A_0 \to B_0$ is a subobject of $f : A \to B$ if the following diagram commutes

$$\begin{tikzcd}
A_0 \arrow[r, hook] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
B_0 \arrow[r, hook] & B
\end{tikzcd} \tag{4.1}$$

where the horizontal maps are inclusions. We will denote this fact by $f_0 \subseteq f$. Given $f : A \to B$ we will write $|f|$ for the number of nondegenerate simplices of $A \vee B$ and will say that f is finite if $|f|$ is. To make the argument easier to follow, through Lemma 4.4 we will refer to elements of $\mathcal{S}_*^2$ as *objects* and to diagrams like (4.1) as *maps*.

Lemma 4.2. *Let $f \subseteq h$ be objects in $\mathcal{S}_*^2$, let g in $\mathcal{S}_*^2$ be compact and let $\alpha \in \hom_*(f, g)$. If for every finite subobject $k \subseteq h$ the map α extends to $f \cup k$ then α extends to h.*

Proof. Let t be in $\mathcal{S}_*^2$ such that $f \subseteq t \subseteq h$. Let $r : \hom_*(t, g) \to \hom_*(f, g)$ be the restriction map. Define $E(t)$ as $r^{-1}(\{\alpha\})$ that is the set of all extensions of α to t. Since r is a continuous map between compact spaces we see that $E(t)$ is empty or compact. Also $E(h) = \lim E(f \cup k)$, where the limit is taken over all finite subobjects $k \subseteq h$. Since the limit is directed and the $E(f \cup k)$ are compact (nonempty by assumption) we see that $E(h)$ is nonempty. □

Directly from Lemma 4.2 we obtain the following statement.

Lemma 4.3. *Given* f *and* g *in* $\mathcal{S}^2_*$ *with* g *compact there is a cardinal number* $\tau = \tau(f, g)$ *such that for any* h *in* $\mathcal{S}^2_*$ *with* $f \subseteq h$ *there is* k *in* $\mathcal{S}^2_*$ *such that* $f \subseteq k \subseteq h$ *and* $|k| \le \tau$ *and if* $\alpha : f \to g$ *extends to* $\alpha_k : k \to g$ *then it extends to* $\alpha_h : h \to g$.

Proof. For each $\alpha : f \to g$ which does not factor as $f \hookrightarrow h \to g$ Lemma 4.2 gives us a finite object k_α in $\mathcal{S}^2_*$ such that α does not factor as $f \hookrightarrow f \cup k_\alpha \to g$. We can take $k = f \cup \bigcup_\alpha k_\alpha$. Since each k_α is finite and the number of possible maps α depends only on f and g we see that there is an upper bound for the cardinality of k which depends only on f and g. □

The role of this Lemma is following. We think of f and g as fixed and of h as uncontrollably big. We want the obstruction to extending a map from f to h to be detected on some k whose cardinality we can control.

Lemma 4.4. *Given* f *and* g *in* $\mathcal{S}^2_*$ *with* g *compact there is a cardinal number* $\delta = \delta(f, g)$ *such that for any* h *in* $\mathcal{S}^2_*$ *with* $f \subseteq h$ *there is* k *in* $\mathcal{S}^2_*$ *such that* $f \subseteq k \subseteq h$ *and* $|k| \le \delta$ *and the restriction map* $\hom_*(h, g) \twoheadrightarrow \hom_*(k, g)$ *is an epimorphism.*

Proof. The object k is constructed as a union of an ascending chain $f = k_0 \subseteq k_1 \subseteq ... \subseteq k_{n-1} \subseteq k_n \subseteq ....$ This chain is built by induction on n. Given k_n we use Lemma 4.3 to choose k_{n+1} so that $k_n \subseteq k_{n+1} \subseteq h$ and if a map $k_n \to g$ extends to k_{n+1} then it extends to h.

Given $\alpha : k \to g$ we need to show that we can extend α to $\tilde{\alpha} : h \to g$. By the construction of k there are maps $\alpha_n : h \to g$ such that $\alpha_n|_{k_n} = \alpha|_{k_n}$. Since by Lemma 3.4 $\hom_*(h, g)$ is compact we can take $\tilde{\alpha}$ to be an accumulation point of the set $\{\alpha_n\}$.

To prove that $\tilde{\alpha}|_k = \alpha$ it is enough to show that $\tilde{\alpha}|_{k_n} = \alpha|_{k_n}$ for all n. This is the case since the sequence $\alpha_i|_{k_n} \in \hom_*(k_n, g)$ converges to $\alpha|_{k_n}$, it is actually constant for $i \ge n$, and the restriction map $\hom_*(h, g) \to \hom_*(k_n, g)$ is continuous.

□

Lemma 4.5. *Let h, p and g be in $\mathcal{S}^2_*$, g be compact, p be a retract in $\mathcal{S}^2_*$ of g and h have the left lifting property with respect to p. There is a cardinal $\gamma = \gamma(g)$ such that h is a colimit of subobjects h_α such that each h_α has the left lifting property with respect to p and $|h_\alpha| \leq \gamma$.*

Proof. We can write h as $h = \mathrm{colim} h_\alpha$ where each h_α is finite. Inductively we replace h_α with objects $h_{*\alpha}$ that have the left lifting property with respect to p. We start with the trivial object in $\mathcal{S}^2_*$, a map between spaces consisting of a basepoint only, which need not be replaced. Suppose that for some α_0 all subobjects of h_{α_0} have been replaced. Let $h' = h_{\alpha_0} \cup \bigcup_{\alpha < \alpha_0} h_{*\alpha}$. Lemma 4.4 gives us a factorization

$$h' \hookrightarrow h_{*\alpha_0} \hookrightarrow h$$

such that the restriction map

$$\hom_*(h, g) \twoheadrightarrow \hom_*(h_{*\alpha_0}, g) \tag{4.6}$$

is an epimorphism. We want to show that $h_{*\alpha_0}$ has the left lifting property with respect to p. Consider a diagram

$$\begin{tikzcd}
h_{*\alpha_0} \arrow[r] \arrow[d, hook] & p \arrow[d] \\
h \arrow[r, "s"] & g \arrow[u, bend right]
\end{tikzcd}$$

where the map s exists by (4.6). Since by assumption h has the left lifting property with respect to p and any map from $h_{*\alpha_0}$ to p factors through h we obtain the left lifting property for $h_{*\alpha_0}$ with respect to p. We see that $|h_{*\alpha_0}|$ depends only on g, on $h_{*\alpha}$ for $\alpha < \alpha_0$ and on the bounds $\delta(h_{*\alpha}, g)$ from Lemma 4.4. □

We are ready to prove the main theorem of this chapter.

Theorem 4.7. *Let Z be a fibrant simplicial set and βZ a simplicial compact space. If Z is a retract of βZ then there exists a map f such that L_f is a localization at Z.*

Proof. To use Lemma 2.7 we consider maps

$$p : \mathrm{map}_*(\Delta^n_+, Z) \to \mathrm{map}_*(\partial\Delta^n_+, Z)$$

and

$$g : \mathrm{map}_*(\Delta^n_+, \beta Z) \to \mathrm{map}_*(\partial\Delta^n_+, \beta Z).$$

We observe that p is a retract of g and g is compact by Corollary 3.6. By Lemma 2.7 an injective map h is a Z-equivalence if and only if it has the left lifting property with respect to p. By Lemma 4.4 there is a cardinal $\gamma = \gamma(g)$ such that any injective Z-equivalence h is a colimit of injective Z-equivalences whose cardinalities do not exceed γ. Since this is a directed colimit of cofibrations it is equivalent to a homotopy colimit. By Lemma 2.6 we can take f to be a wedge of all Z-equivalences whose cardinality does not exceed γ. □

Considering a fibrant simplicial set Z which is a retract of a simplicial compact space βZ might look artificial. To avoid this impression we provide the following.

Example 4.8. Let $n > 0$, $Z = K(\mathbb{Q}, n)$ and $\beta Z = K(S^1, n)$. As a model of $K(S^1, n)$ we use the one described in 1.2 of [4]; $K(S^1, n)_t$ is a product of $\binom{t}{n}$ copies of S^1, faces and degeneracies are given by projections and group operations hence are continuous. This model of $K(S^1, n)$ is a simplicial compact space. It has a homotopy type of an Eilenberg-Mac Lane space for S^1 viewed as a discrete group. The group S^1 is a direct sum of $\mathbb{Q}/\mathbb{Z}$ and a rational vector space hence $\mathbb{Q}$ is a retract of S^1 and so Z is a retract of βZ. Corollary 3.11 implies that Z has no structure of a fibrant simplicial compact space since $\pi_n Z = \mathbb{Q}$ is infinite countable.

CHAPTER 5

COHOMOLOGICAL LOCALIZATIONS OF SPACES

As an immediate application of Theorem 4.7 we obtain the existence of certain cohomological localizations.

Theorem 5.1. *Let* h^* *be a cohomology theory represented by an* Ω*-spectrum* $\{\underline{h}_n\}$*. If each* $\underline{h}_n$ *is equivalent to a fibrant simplicial set which is a retract of a simplicial compact space then there exists a map* f *such that* L_f*-equivalences and* h^**-equivalences coincide.*

Proof. Let $Z = \prod \underline{h}_n$ and use Theorem 4.7. □

CHAPTER 6

PROFINITE COMPLETION

In this chapter we prove theorem 6.2.

The profinite completion was introduced by Sullivan in Section 3 of [20]. Here it will be convenient to describe it (see [20] after Proposition 3.3) as a coaugmented functor

$$X \to \hat{X} = \text{holim } F$$

where the homotopy limit is taken over the category $(X \downarrow \mathcal{F})$ whose objects are maps $X \to F$ with F connected and $\pi_q F$ finite for all $q > 0$. The morphisms are homotopy commutative diagrams

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ F_1 \arrow[rr] & & F_2 \end{tikzcd} \tag{6.1}$$

This homotopy limit is obtained via the Brown representability theorem and is well defined since the category $(X \downarrow \mathcal{F})$ is equivalent to a small category. The functor $F : (X \downarrow \mathcal{F}) \to \mathcal{S}_*$ takes an object $X \to F_0$ to the space F_0.

By an idempotent approximation to $X \to \hat{X}$ we understand the terminal localization L among those that admit the factorization below.

$$X \to LX \to \hat{X}$$

Theorem 6.2. *There exists an idempotent approximation to the profinite completion. More precisely, there is the terminal localization among localizations L that admit the following factorization.*

$$X \to LX \to \hat{X}$$

Proof. For each homotopy class of fibrant connected simplicial sets with $\pi_q F$ finite for all $q > 0$ choose a representative F. We can assume that these representatives are minimal fibrant complexes. Such simplicial sets are finite in each dimension hence compact. Dimension-wise finiteness implies also that these representatives form a set hence we can form a product $Z = \prod F$ over all F's as above. The space Z is a fibrant simplicial compact space. The localization L_Z exists by Theorem 4.7. We note also that if F_0 is connected with $\pi_q F_0$ finite for $q > 0$ then F_0 is Z-local. Let $r : Z \to F_0 \hookrightarrow Z$ be the retraction onto the axis that corresponds to F_0. We see that $F_0 \simeq \mathrm{holim}(... \xrightarrow{r} Z \xrightarrow{r} Z)$ hence it is Z-local.

We claim that the categories $(X \downarrow \mathcal{F})$ and $(L_Z X \downarrow \mathcal{F})$ are equivalent. Since the profinite completion preserves weak equivalences we can assume that the localization map $\eta_X : X \hookrightarrow L_Z X$ is a cofibration. Let $A : (L_Z X \downarrow \mathcal{F}) \to (X \downarrow \mathcal{F})$ be the functor defined as $A(i) = i \circ \eta_X$. Let $B : (X \downarrow \mathcal{F}) \to (L_Z X \downarrow \mathcal{F})$ be a functorial localization over $\mathcal{F}$ constructed as follows. For an object k in $(X \downarrow \mathcal{F})$ we choose $B(k)$ in $(L_Z X \downarrow \mathcal{F})$ so that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[rr, hook, "\eta_X"] \arrow[dr, "k"'] & & L_Z X \arrow[dl, "B(k)"] \\
& F_0 &
\end{tikzcd}$$

Such $B(k)$ exists because F_0 is Z-local. The choice of $B(k)$ is unique only up to homotopy. Still no matter what choices we make the resulting functor B is well defined because maps between objects in $(L_Z X \downarrow \mathcal{F})$ and $(X \downarrow \mathcal{F})$ as in (6.1) are determined by the maps between the underlying spaces F_1 and F_2 which are not affected by our functors. Also we did not destroy the homotopy commutativity of the diagram (6.1). The composition AB is an identity on $(X \downarrow \mathcal{F})$. We also have a natural equivalence $Id_{(L_Z X \downarrow \mathcal{F})} \xrightarrow{\simeq} BA$ given by

$$\begin{tikzcd}
& L_Z X \arrow[dl, "i"'] \arrow[dr, "B(i)"] & \\
F_0 \arrow[rr, "id"] & & F_0.
\end{tikzcd}$$

The equivalence of categories gives us an equivalence of homotopy inverse limits $\hat{X} \simeq (L_Z X)\hat{}$ which leads to the factorization we were looking for:

$$X \to L_Z X \to \hat{X} \tag{6.3}$$

It remains to show that L_Z is the terminal localization which admits factorization (6.3). Suppose that a localization T also admits (6.3). Since profinite completion is idempotent on finite spaces F as above we have

$$F \to TF \to \hat{F} \simeq F$$

so F is a homotopy retract of TF hence T-local. This means that the space Z is T-local hence by the definition of L_Z we have $T \leq L_Z$. $\square$

CHAPTER 7

RELATIVE PHANTOM MAPS AND κ-PHANTOM MAPS

We will use the relative phantom maps to identify certain homotopy inverse limits. In this chapter we introduce the concept and essential properties. As far as we know this is the first place where the relative phantom maps are considered. The concept is purely unstable since in a triangulated category a relative map is phantom if and only if the induced map of cofibers is phantom. On the other side we think that even if one would like to translate the arguments, presented in subsequent chapters, to the stable category the line of reasoning would be more clear when presented in the language of relative phantom maps.

We will work in the *category of pairs* of pointed simplicial sets $\mathcal{S}_*^2$ whose objects are maps between simplicial sets and whose morphisms are commutative squares

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "i"'] & X_0 \arrow[d, "j"] \\ A \arrow[r, "f"] & X \end{tikzcd}$$

We are interested in the *homotopy category of pairs* $ho\mathcal{S}_*^2$ which can be described as follows (see [3] A3). A map from i to j as above is a *weak equivalence* if f_0 and f are equivalences. Whenever we use notation (A, A_0) for an object $i : A_0 \to A$ it means that either we assume that i is a cofibration (i.e. an injection) or we replace i with an equivalent cofibration. A map $(A, A_0) \to (X, X_0)$ will be called *trivial* if for some K it factors as $(A, A_0) \to (K, K) \to (X, X_0)$. Equivalently the map is trivial if there exists a lift

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

It will be called *homotopically trivial* if it is homotopic to a trivial map. We will say that an object has *relative cardinality* not bigger than κ if it is equivalent to a pair (F, F_0) such that $F_0 \hookrightarrow F$ is a cofibration and there are no more than κ nondegenerate simplices in $F \smallsetminus F_0$. Similarly we will say that (F, F_0) has *relative dimension* not bigger than n if all nondegenerate simplices in $F \smallsetminus F_0$ have dimension not bigger than n.

Definition 7.1. A map $(X, X_0) \to (Z, Z_0)$ between pairs of simplicial sets is *phantom* if any composition

$$(F, F_0) \to (X, X_0) \to (Z, Z_0)$$

with (F, F_0) finite is homotopically trivial.

Note that one obtains an equivalent definition if the condition that (F, F_0) is finite is replaced with the one saying that F is finite. This is so because the homotopy is allowed to move only the simplices in $F \smallsetminus F_0$ which in both cases form a finite set.

The following generalizes Theorem 3.20(i) in [16].

Proposition 7.2. *If* $f : (X, X_0) \to (Y, Y_0)$ *is a phantom map with* (Y, Y_0) *fibrant and* $g : (Y, Y_0) \to (Z, Z_0)$ *is a map with a compact (not necessarily fibrant) target then the composition* gf *is homotopically trivial.*

Proof. We have $(X, X_0) = \operatorname{colim}(F_\alpha, F_{0\alpha})$ where the colimit ranges over all finite $(F_\alpha, F_{0\alpha}) \subseteq (X, X_0)$. The composition gf and inclusions of $(F_\alpha, F_{0\alpha})$ to (X, X_0) induce maps $k_\alpha : (F_\alpha, F_{0\alpha}) \to (Z, Z_0)$. Let $C(F_\alpha, F_{0\alpha})$ be the set of contractions of k_α, that is the set of maps

$$h : (F_\alpha \wedge I_+, F_{0\alpha} \wedge I_+) \to (Z, Z_0)$$

such that $h|_{F_\alpha \times \{0\}} = k_\alpha$ and $h(F_\alpha \times \{1\}) \subseteq Z_0$. Since f is phantom the sets $\mathrm{C}(F_\alpha, F_{0\alpha})$ are nonempty, and compact by Lemma 3.7. Therefore the set of contractions of gf, namely $\mathrm{C}(X, X_0) = \lim \mathrm{C}(F_\alpha, F_{0\alpha})$, is nonempty so gf is homotopically trivial as required. $\square$

Recall that the Čech-Stone compactification β is an idempotent functor in the category of completely regular (i.e. $T_{3\frac{1}{2}}$) topological spaces. There is a natural map $c : X \to \beta X$ which is initial among all continuous maps from X to a compact space. For a simplicial space X by βX we mean the levelwise compactification. If X is a simplicial set then βX stands for the compactification of the corresponding simplicial discrete space. Note that even if X is fibrant we see no reason for βX to be fibrant. Immediately from Proposition 7.2 we obtain the following.

Corollary 7.3. *If $(X, X_0) \to (Y, Y_0)$ is a phantom map with (Y, Y_0) fibrant then the composition $(X, X_0) \to (Y, Y_0) \to (\beta Y, \beta Y_0)$ is homotopically trivial.*

We observe that the compactification β of fibrant simplicial sets is well defined on the homotopy category.

Proposition 7.4. *If $f : X \to Y$ and $g : X \to Y$ are homotopic maps between simplicial sets (i.e. simplicial discrete spaces) and Y is fibrant then the Čech-Stone compactifications βf and βg are homotopic.*

Proof. The maps f and g are homotopic if and only if there exists a homotopy between f and g, that is a set of maps $h_i : X_n \to Y_{n+1}$, $0 \leq i \leq n$ for each $n \geq 0$ such that appropriate diagrams commute (see [15] 5.1). By functoriality of β the compactifications βh_i form a homotopy between βf and βg. □

Corollary 7.5. *If simplicial sets X and Y are fibrant and $X \simeq Y$ then $\beta X \simeq \beta Y$.*

Definition 7.6. We will say that a map $(X, X_0) \to (Z, Z_0)$ is *skeleton-phantom* if the composite $\mathrm{sk}_n(X, X_0) \to (X, X_0) \to (Z, Z_0)$ is homotopically trivial for any $n \geq 0$.

The symbol $\mathrm{sk}_n(X, X_0)$ denotes the *n-skeleton* of (X, X_0) that is the simplicial pair $(X^{(n)}, X_0)$ such that $X^{(n)}$ is the union of X_0 and the usual n-skeleton of X. Obviously any skeleton-phantom map is phantom. The converse is not true but still we have the following proposition.

Proposition 7.7. *A composite of n phantom maps between fibrant pairs*

$$(X^0, A^0) \xrightarrow{f_0} (X^1, A^1) \xrightarrow{f_1} ... \xrightarrow{f_{n-1}} (X^n, A^n)$$

is homotopically trivial on the $(n-1)$*-skeleton of the source.*

Proof. We will show a slightly stronger result that we can modify up to homotopy the maps f_0, ..., f_{n-1} so that their composition is actually trivial on the $(n-1)$-skeleton. Since f_0 is phantom it is homotopically trivial on each vertex hence on the zero skeleton of (X^0, A^0). We can modify it up to homotopy so that it is actually trivial on $\mathrm{sk}_0(X^0, A^0)$.

We proceed by induction. Suppose that for some k the composite $g_{k-1} = f_{k-1} \circ ... \circ f_1 \circ f_0$ is trivial on the $(k-1)$-skeleton. We want to modify up to homotopy f_k so that the composition $f_k \circ g_{k-1}$ is trivial on the k-skeleton. A k-simplex in $X^0 \smallsetminus A^0$ is sent by g_{k-1} to a k-simplex in $X^k \smallsetminus A^k$ whose boundary is in A^k. Let x be a nondegenerate k-simplex in $X^k \smallsetminus A^k$ such that its boundary ∂x is in A^k. Let D_x be the simplicial subset of X^k generated by A^k and x. Let $g_x : (D_x, A^k) \to (X^{k+1}, A^{k+1})$ be the restriction of f_k to (D_x, A^k). Since f_k is phantom we can choose a homotopy h_x relative A^k between g_x and a trivial map. For k-simplices x and y as above the corresponding homotopies h_x and h_y are fixed on $D_x \cap D_y \subseteq A^k$. Let $D = \bigcup D_x$ and $h = \bigcup h_x$ be the unions taken over all k-simplices with boundary in A^k. Now h is a homotopy between the restriction of f_k to (D, A^k) and a trivial map from (D, A^k) to (X^{k+1}, A^{k+1}). We can modify f_k within its homotopy class so that $f_k(D) \subseteq A^{k+1}$. Our supposition that $g_{k-1}(\mathrm{sk}_{k-1}(X^0, A^0)) \subseteq (A^k, A^k)$ and the observation that $g_{k-1}(\mathrm{sk}_k(X^0, A^0)) \subseteq (D, A^k)$ imply that

$$g_k(\mathrm{sk}_k(X^0, A^0)) = f_k \circ g_{k-1}(\mathrm{sk}_k(X^0, A^0)) \subseteq f_k((D, A^k)) \subseteq (A^{k+1}, A^{k+1})$$

which completes the inductive step. □

Corollary 7.8. *If a given map f factors as a composite of n phantom maps for all* $n \geq 1$ *then f is skeleton-phantom.*

We prepare to show in Proposition 7.12 that a composite of two relative skeleton-phantom maps is trivial. These results generalize the beginning of Section 3 in [16]. All the simplicial sets are assumed to be fibrant.

We start by showing that for any pair (X, A) there is a weakly initial skeleton-phantom map

$$\delta : (X, A) \to (\tilde{X}, \tilde{A})$$

from (X, A). By "weakly initial" we mean that any other skeleton-phantom map out of (X, A) factors through δ, but the uniqueness is not required. We will use the symbol $[m, n]$ to denote the simplicial set with one vertex for each integer k, $m \leq k \leq n$, one nondegenerate 1-simplex for each segment $[k-1, k]$, $m < k \leq n$ and obvious face and degeneracy maps. The symbol $[0, \infty)$ will denote $\mathrm{colim}_n[0, n]$ with the obvious inclusions. We define

$$\tilde{X} = \bigcup_{n \geq 1} \mathrm{sk}_n(X, A) \wedge [n-1, n]_+ / \sim .$$

where each $X_n \times \{n\}$ is identified with its image in $X_{n+1} \times \{n\}$. The projection

$$p : (\tilde{X}, A \wedge [0, \infty)_+) \to (X, A)$$

is a relative homotopy equivalence. We define a subcomplex $\tilde{A}$ of $\tilde{X}$ as

$$\tilde{A} = A \wedge [0, \infty)_+ \cup \bigcup_{n \geq 0} \mathrm{sk}_n(X, A) \times \{n\}$$

The pair $(\tilde{X}, \tilde{A})$ can be visualized as

where $\tilde{A}$ is the thick part. Let i be a representative of the homotopy inverse to the projection p above. In particular we need to take fibrant replacements of the pairs $(\tilde{X}, A \wedge [0, \infty)_+)$ and $(\tilde{X}, \tilde{A})$. We define δ as a composite of i and the obvious inclusion k

$$\begin{tikzcd}
(X, A) \arrow[r, "i"] \arrow[rr, bend left, "\delta"] & (\tilde{X}, A \wedge [0, \infty)_+) \arrow[r, hook, "k"] & (\tilde{X}, \tilde{A})
\end{tikzcd}$$

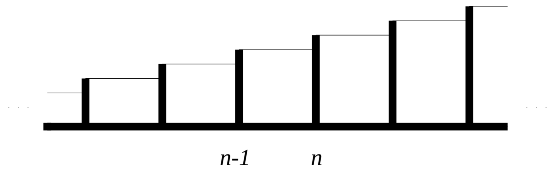

Figure 7.1: $(\tilde{X}, \tilde{A})$

The following two Lemmas show that the map δ constructed above is a weakly universal relative skeleton-phantom map.

Lemma 7.9. *The map δ constructed above is a relative skeleton-phantom map.*

Proof. Consider a composite

$$\begin{tikzcd}
\mathrm{sk}_n(X,A) \arrow[r, hook, "j"] & (X,A) \arrow[r, "i"] \arrow[rr, bend left, "\delta"] & (\tilde{X}, A \wedge [0,\infty)_+) \arrow[r, hook, "k"] & (\tilde{X}, \tilde{A})\,.
\end{tikzcd}$$

The composite $i \circ j$ is homotopic to the inclusion

$$\mathrm{sk}_n(X,A) \cong \mathrm{sk}_n(X,A) \times \{n\} \hookrightarrow (\tilde{X}, A \wedge [0,\infty)_+)$$

whose composite with k is trivial since all the domain goes to $\tilde{A}$. □

Lemma 7.10. *Any relative skeleton-phantom map out of (X, A) factors through δ.*

Proof. Without loss of generality we can replace (X, A) with the homotopy equivalent pair $(\tilde{X}, A \wedge [0,\infty)_+)$. Let $f : (\tilde{X}, A \wedge [0,\infty)_+) \to (Y, B)$ be a relative skeleton-phantom map. Fix an integer $n > 0$. A restriction of f to a finite dimensional pair

$$(\mathrm{sk}_n(X,A) \times \{n\}) \cup ((A,A) \wedge [0,\infty)_+) \tag{7.11}$$

is, by assumption, homotopic relative $A \wedge [0,\infty)_+$ to a trivial map. Since all the (7.11) are pairwise disjoint outside of $(A,A) \wedge [0,\infty)_+$ we see that the restriction of f to their union $(\tilde{A}, A \wedge [0,\infty)_+)$ is homotopic to a trivial map, that is the one which takes $\tilde{A}$ to B. Since $(\tilde{A}, A \wedge [0,\infty)_+) \subseteq (\tilde{X}, A \wedge_+ [0,\infty))$ is a cofibration we can replace f with a homotopic map $\tilde{f}$ such that $\tilde{f}(\tilde{A}) \subseteq B$. The map $\tilde{f}$ determines the required factorization. □

The construction of a weakly universal phantom map out of a pair allows us to prove the following result.

Proposition 7.12. *If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ are two skeleton-phantom maps then the composite $gf : (X, A) \to (Z, C)$ is homotopically trivial.*

Proof. As a consequence of Lemma 7.10 we obtain the following commutative diagram:

$$\begin{tikzcd}
(X,A) \arrow[r, "f"] \arrow[d, "\delta"'] & (Y,B) \arrow[r, "g"] & (Z,C) \\
(\tilde{X},\tilde{A}) \arrow[ur, "\tilde{f}"'] & &
\end{tikzcd}$$

Let $(\tilde{X}_n, \tilde{A}) = \mathrm{sk}_n(X, A) \wedge [n-1, n]_+ \cup (\tilde{A}, \tilde{A})$. In other words $\tilde{A}$ is the thick part and $\tilde{X}_n$ is the thick and the bricked part of the telescope below representing $(\tilde{X}, \tilde{A})$.

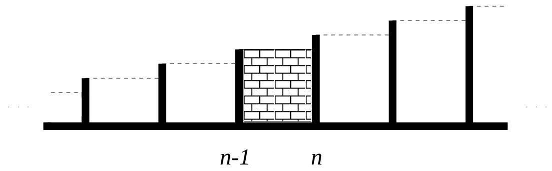

Figure 7.2: $\tilde{X}_n$ in $(\tilde{X}, \tilde{A})$

We see that $(\tilde{X}, \tilde{A}) = \bigcup_{n \geq 1} (\tilde{X}_n, \tilde{A})$. For $n \neq m$ we have $\tilde{X}_n \cap \tilde{X}_m \subseteq \tilde{A}$. Also the relative dimension of $(\tilde{X}_n, \tilde{A})$ is not bigger than n. Since g is skeleton-phantom the composite

$$(\tilde{X}_n, \tilde{A}) \hookrightarrow (\tilde{X}, \tilde{A}) \xrightarrow{\tilde{f}} (Y, B) \xrightarrow{g} (Z, C)$$

is homotopic relative $\tilde{A}$ to a trivial map. Since as we observed different $(\tilde{X}_n, \tilde{A})$'s are disjoint outside of $\tilde{A}$ the composition $g\tilde{f}$ is homotopic relative $\tilde{A}$ to a trivial map. □

Localization at a space is related to κ-phantom maps where κ is an infinite cardinal number (see Chapter 7 in [14]).

Definition 7.13. A map $(X, X_0) \to (Z, Z_0)$ between pairs of simplicial sets is *κ-phantom* if a composition

$$(F, F_0) \to (X, X_0) \to (Z, Z_0)$$

is homotopically trivial whenever the cardinality of (F, F_0) is less than κ.

Obviously any κ-phantom map is phantom. The following example shows that there are arbitrarily large cardinals κ with a nontrivial κ-phantom map. An infinite cardinal number κ is called *regular* if whenever $\kappa = \bigcup_{i \in I} \lambda_i$ then either $|I| = \kappa$ or for some i we have $\lambda_i = \kappa$. Since every successor cardinal is regular we have arbitrarily large regular cardinals.

Example 7.14. Let κ be a regular cardinal. Let $\mathbb{Z}^\kappa$ be the product of κ copies of integers. Let $\mathbb{Z}^{<\kappa}$ be the subgroup of $\mathbb{Z}^\kappa$ consisting of elements which only have fewer than κ nonzero coordinates. For $n \geq 2$ consider the following fibration.

$$K(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, n-1) \xrightarrow{\varphi} K(\mathbb{Z}^{<\kappa}, n) \to K(\mathbb{Z}^\kappa, n)$$

We claim that φ is a nontrivial κ-phantom map.

Proof. First we show that φ is κ-phantom. We will do it in case when κ is uncountable. It is enough to show that any diagram

$$\begin{tikzcd}
K(\mathbb{Z}^\kappa, n-1) \arrow[r] & K(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, n-1) \arrow[r, "\varphi"] & K(\mathbb{Z}^{<\kappa}, n) \\
& A \arrow[u, "k"] \arrow[ul, dashed, "r"] &
\end{tikzcd}$$

with $|A| < \kappa$ admits the homotopy lift r. Since k factors up to homotopy through $A/A^{(n-2)}$ where $A^{(n-2)}$ is the $n-2$ skeleton of A we may assume that $A^{(n-2)} = *$.

Since we assumed that κ is uncountable $|A| < \kappa$ implies $|\mathrm{Ex}^\infty A| < \kappa$ so we may assume that A is fibrant. Then the cardinality of $\pi_{n-1}A$ is less than κ. Lemma

3.1 in [9] tells us that there is a lift in the category of groups as indicated in the diagram below.

$$\begin{tikzcd}
\mathbb{Z}^{\kappa} \arrow[rr] & & \mathbb{Z}^{\kappa}/\mathbb{Z}^{<\kappa} \\
& \pi_{n-1}A \arrow[ul, dashed] \arrow[ur] &
\end{tikzcd}$$

This means the homotopy lift r exists on the n skeleton $A^{(n)}$. Vanishing of the higher obstructions allows us to extend this lift to whole the space A. This proves that φ is κ-phantom.

We will show that φ is not trivial. If $\varphi \simeq *$ then we would have

$$K(\mathbb{Z}^{\kappa}, n-1) \simeq K(\mathbb{Z}^{<\kappa}, n-1) \times K(\mathbb{Z}^{\kappa}/\mathbb{Z}^{<\kappa}, n-1)$$

so the epimorphism $\mathbb{Z}^{\kappa} \twoheadrightarrow \mathbb{Z}^{\kappa}/\mathbb{Z}^{<\kappa}$ would split. In a personal communication the author learned from O'Neil that this is impossible because of a combination of set theoretic and group theoretic arguments. $\square$

CHAPTER 8

RELATIVE PHANTOM MAPS OVER LONG TOWERS

In this chapter we associate a relative phantom map to a given map under X. We conjecture (in 8.3) that given a long enough tower under X we can produce two such relative phantom maps which are compatible in an appropriate sense.

We are interested in such constructions for the following reason. Consider a relative phantom map $(R_1, K) \to (R_0, K)$ as the square below.

$$\begin{tikzcd}
K \arrow[r, equal] \arrow[d] & K \arrow[d] \\
R_1 \arrow[r] & R_0
\end{tikzcd}$$

In our applications the map $R_1 \to R_0$ is built by means of homotopy limits while the simplicial set K is built by means of homotopy colimits. If we find a way to compose such maps we can use Corollary 7.8 and Proposition 7.12 to show that the composition

$$(R_{\omega+\omega}, K) \to ... \to (R_\omega, K) \to ... \to (R_1, K) \to (R_0, K)$$

is trivial hence K is a retract of $R_{\omega+\omega}$. This is the strategy of our attempt to show that localization at certain spaces can be built by means of homotopy colimits as well as homotopy limits.

Lemma 8.1. *Given a map $X \to R_1 \to R_0$ there is a relative phantom map*

$(R_1, K) \to (R_0, K)$ under X, that is a diagram

$$\begin{tikzcd}
 & X \arrow[dl, hook'] \arrow[d, hook] \\
K \arrow[r, equal] \arrow[d] & K \arrow[d] \\
R_1 \arrow[r] & R_0
\end{tikzcd}$$

such that the cardinality of K depends only on the map $X \to R_0$.

Proof. Let us note that if we do not impose any restrictions on K we can take $K = R_1$.

We define $t : K \to R_1$ by induction over ω, the order of the natural numbers. Let $K_0 = X$. If K_i is defined let K_{i+1} be the Kan Ex^{∞} functor of the homotopy pushout of the following diagram.

$$\begin{tikzcd}
\bigvee_f A \arrow[r] \arrow[d, hook] & K_i \arrow[d, hook] \arrow[rrd, bend left, "t_i"] & & \\
\bigvee_f B \arrow[r] \arrow[rrr, bend right] & P \arrow[r, hook] & \mathrm{Ex}^{\infty} P = K_{i+1} \arrow[r, "t_{i+1}"] & R_1
\end{tikzcd} \tag{8.2}$$

The wedge is taken over those maps $f : (B, A) \to (R_0, K_i)$ for which $|A|$ and $|B|$ are finite and the diagram

$$\begin{tikzcd}
A \arrow[rr, "f_A"] \arrow[d, hook] & & K_i \arrow[d] \arrow[dl, "t_i"'] \\
B \arrow[rr, "f_B"] \arrow[r, dashed, "r"'] & R_1 \arrow[r, "p"'] & R_0
\end{tikzcd}$$

admits the homotopy lift r. Those lifts are used to define $t_{i+1} : K_{i+1} \to R_1$ as in (8.2). We define K as $\mathrm{colim}_{i<\omega} K_i$ and $t = \mathrm{colim}_{i<\omega} t_i$.

The resulting map $(R_1, K) \to (R_0, K)$ is phantom since any map $A \to K$ with A finite factors through some $K_i \subseteq K$ so that for B finite we have the following diagram

$$\begin{tikzcd}
A \arrow[r, "f_A"] \arrow[d, hook] & K_i & \subseteq & K_{i+1} & \subseteq & K \arrow[dllll] \arrow[d] \\
B \arrow[r, "f_B"'] \arrow[urrr, dashed, "k"] & R_1 \arrow[rrrr, "p"] & & & & R_0
\end{tikzcd}$$

where the lift $B \dashrightarrow K_{i+1}$ exists by the construction of K. ☐

Conjecture 8.3. *For any map $X \to R_0$ there is an ordinal μ such that for any tower under X indexed over μ as on the diagram below*

$$\begin{tikzcd}
& & & & X \arrow[dllll] \arrow[dll] \arrow[d] \\
R_\mu \arrow[r] & \cdots \arrow[r] & R_i \arrow[r] & \cdots \arrow[r] & R_0
\end{tikzcd}$$

there is an ordinal $\mu_0 < \mu$ and a factorization

$$\begin{tikzcd}
X \arrow[r, hook] & K \arrow[dl] \arrow[d] \\
R_{\mu_0+1} \arrow[r] \arrow[dr] & R_{\mu_0} \arrow[d] \\
& R_0
\end{tikzcd}$$

such that the induced map $(R_{\mu_0}, K) \to (R_0, K)$ is relative phantom.

By Lemma 8.1 we have plenty of phantom maps $(R_{\mu_0}, K) \to (R_0, K)$ so the Conjecture states that for one of them the map $K \to R_{\mu_0}$ lifts to R_{μ_0+1}.

We think it should be possible to prove this conjecture at least in the case when the words “any tower” are replaced with a phrase “a tower as described in Definition 9.6”.

CHAPTER 9

A LONG LOCALIZATION TOWER

This chapter is written assuming Conjecture 8.3. We deduce that the image in $ho\mathcal{S}_*$ of the functor of a localization at a simplicial set Z, as in Theorem 4.7, is the smallest class which contains Z and is closed under weak equivalences and arbitrary homotopy inverse limits. This is related to a result by Dror and Dwyer who showed in [10] that when $Z = \prod_{n>0} K(R, n)$ where R is a finite field of the form $\mathbb{Z}/p\mathbb{Z}$, p prime then the $H_*(-, R)$-localization has a similar image. Theorem 9.11 would allow to extend this result to, for example, localization at Morava K-theory.

Throughout this chapter we fix a fibrant simplicial set Z which is a retract of a simplicial compact space βZ. By Theorem 4.7 there is a map $f : A \to B$ such that localization at Z is equivalent to the f-localization.

Definition 9.1. Let $\mathcal{L}(Z)$ be the least class that contains Z and is closed under weak equivalences and arbitrary homotopy inverse limits.

By Definition 2.5 (*iii*) localization at Z can be characterized by saying that the map $X \to L_Z X$ is a Z-equivalence and $L_Z X$ is Z-local. Since any R in $\mathcal{L}(Z)$ is Z-local and a cofibration $X \hookrightarrow R$ is a Z-equivalence if and only if it has a left lifting property with respect to the map p as in Lemma 2.7 it is natural to attempt to construct $L_Z X$ by means of homotopy inverse limits as described in Definitions 9.2 and 9.6 below.

Definition 9.2. Given a simplicial set X we construct a square

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d] & S \arrow[d, "p"] \\ R \arrow[r] & T \end{tikzcd} \tag{9.3}$$

as follows. Let $p : S \to T$ be the map

$$p : \prod_{n\geq 0} \prod_{h} \mathrm{map}_*(\Delta^n_+, Z) \to \prod_{n\geq 0} \prod_{h} \mathrm{map}_*(\partial\Delta^n_+, Z) \tag{9.4}$$

where h in the second product runs over all maps $h : X \to \mathrm{map}_*(\Delta^n_+, Z)$. Let d be the map with coordinates the maps h:

$$d : X \to \prod_{n\geq 0} \prod_{h} \mathrm{map}_*(\Delta^n_+, Z) \tag{9.5}$$

Let $R = S$ so that the square closes trivially.

The simplicial sets S, T and R are obviously in $\mathfrak{L}(Z)$, since they are built out of Z by means of homotopy limits, hence they are Z-local.

Since the map d in (9.5) carries information about all the maps from X to $\mathrm{map}_*(\Delta^n_+, Z)$ we see from Lemma 2.7 that the map $X \to R$ is a Z-equivalence if and only if any map $R \to T$ under X lifts to S. Since there is no reason to expect that this would happen in the setting of Definition 9.2 it is natural to introduce the following construction.

Definition 9.6. Given a square (9.3) Let $R_0 = R$ and assume that μ is an ordinal assigned to the map $X \to R_0$ as in Conjecture 8.3. We construct a tower under X indexed over μ as follows. If R_α is defined let $R_{\alpha+1}$ be the homotopy limit of the diagram under X

$$\begin{tikzcd}
X \arrow[drr, "d"] \arrow[dr, "d_{\alpha+1}"] \arrow[ddr, "d_\alpha"'] & & \\
& R_{\alpha+1} \arrow[r] \arrow[d] & S \arrow[d] \\
& R_\alpha \arrow[r, shift left, "g_i"] \arrow[r, shift right] \arrow[r, phantom, "\cdots"] & T
\end{tikzcd}$$

where the maps g_i run over all homotopy classes of maps under X. If α is a limit ordinal and R_i are defined for $i < \alpha$ let $R_\alpha = \mathrm{holim}_{i<\alpha} R_i$ and $d_\alpha = \mathrm{holim}_{i<\alpha} d_i$.

The following Lemma is obvious from the remarks preceding Definitions 9.2 and 9.6.

Lemma 9.7. *If the tower described in Definition 9.6 admits a section under X as below*

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] & \\
R_{\alpha+1} \arrow[rr] & & R_\alpha \arrow[ll, dashed, bend right]
\end{tikzcd}$$

then the map $X \to R_\alpha$ is the localization of X at Z.

Definition 9.8. Let $R_\mu \to ... \to R_1 \to R_0$ be the tower under X constructed in Definition 9.6 and μ_0 be the ordinal number as in Conjecture 8.3. Let $K(X) = K$ and $V(R_0) = R_{\mu_0}$ as in Conjecture 8.3. We obtain a diagram

$$\begin{tikzcd}
& & X \arrow[dl] \arrow[d] \\
& K(X) \arrow[r, equal] \arrow[dl, "s"'] \arrow[d] & K(X) \arrow[d] \\
R_{\mu_0+1} \arrow[r] & V(R_0) \arrow[r] & R_0
\end{tikzcd}$$

where the map $(V(R_0), K(X)) \to (R_0, K(X))$ is phantom.

Lemma 9.9. *Assuming Conjecture 8.3, if a fibrant simplicial set Z is a retract of a simplicial compact space βZ then the map $X \to K(X)$ is a Z-equivalence.*

Proof. We can assume that $X \to K(X)$ is a cofibration. Let p be the map as in (9.4) and d be the map as in (9.5). By the remarks after Definition 9.2 it is enough to show that for any h which closes the following square

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[d] & S \arrow[d, "p"] \\
K(X) \arrow[r, "h"] \arrow[ur, dashed] & T
\end{tikzcd}$$

there is a lift as indicated by the dashed arrow. Let $g : \overline{S} \to \overline{T}$ be a map constructed as p in (9.4) but with βZ instead of Z. The map g is compact and p is a retract, in the category of maps $\mathcal{S}^2_*$, of g. Consider the following diagram

$$\begin{tikzcd}
 & & & & X \arrow[rr] \arrow[dd] & & S \arrow[dl] \arrow[dd, "p"] \\
 & & & & & \overline{S} \arrow[ur, bend right] \arrow[dd, "g"] & \\
 & & K(X) \arrow[rr, equal] \arrow[dl] \arrow[dd, "t"] \arrow[ddll, bend right, "s"'] & & K(X) \arrow[rr, "h"] \arrow[dl, "j"] \arrow[dd] & & T \arrow[dl, "i"] \\
 & \beta K(X) \arrow[rr, equal] \arrow[dd] & & \beta K(X) \arrow[rr, "\overline{h}"] \arrow[dd] & & \overline{T} \arrow[ur, bend right, "r"'] & \\
R_{\mu_0+1} \arrow[rr] & & V(R_0) \arrow[rr] \arrow[dl] \arrow[ur, dashed, "c"] & & R_0 \arrow[dl] & & \\
 & \beta V(R_0) \arrow[rr] & & \beta R_0 & & &
\end{tikzcd}$$

There are two ways of looking at this diagram that should make the arguments presented below more clear. First: the entries in the back plane are simplicial sets while the entries in the front plane are simplicial compact spaces. Second: it may be convenient to look at the vertical maps as objects in $\mathcal{S}^2_*$.

The map s comes from Definition 9.8 and it exists under assumption of Conjecture 8.3. The symbol β denotes the level-wise Čech-Stone compactification described before Corollary 7.3. The map $\overline{h}$ comes from the universal property of Čech-Stone compactification.

The first map of the composition $(V(R_0), K(X)) \to (R_0, K(X)) \to (\beta R_0, \beta K(X))$ is phantom by Definition 9.8 and the target of the second one is compact thus by Corollary 7.3 the composition is homotopically trivial. That means the map $V(R_0) \to \beta R_0$ factors, up to homotopy under $K(X)$, through $\beta K(X)$. We get $h = rih = r\overline{h}j \simeq r\overline{h}ct$ so the map h factors up to homotopy through a map from $V(R_0)$ to T. By Definition 9.6 there is a map $r : R_{\mu_0+1} \to S$ under X such that the composition pr factors through all the maps under X from $V(R_0) = R_{\mu_0}$ to T. This implies that the composition of s and the map $r : R_{\mu_0+1} \to S$ forms a required lift. $\square$

Lemma 9.10. *Under assumptions as in Lemma 9.9, if X is Z-local then the relative map $(V(R_0), X) \to (R_0, X)$ is phantom.*

Proof. We consider the following homotopically commutative diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[ddd, hook] & X \arrow[r, equal] \arrow[d, hook, "\simeq_Z"'] & X \arrow[d, hook, "\simeq_Z"'] & \\
& K(X) \arrow[r, equal] \arrow[d] & K(X) \arrow[d] & \\
& L_ZK(X) \arrow[r, equal] \arrow[d] & L_ZK(X) \arrow[d] \arrow[r, phantom, "\simeq"] & X \arrow[uul, dashed] \\
B \arrow[r] \arrow[uurr, dashed, bend left] & V(R_0) \arrow[r] & R_0 &
\end{tikzcd}$$

Since both $V(R_0)$ and R_0 are Z-local the map from $K(X) = K(X)$ to $V(R_0) \to R_0$ factors through Z-localization of $K(X)$ which, by Lemma 9.9, is homotopically equivalent to X. If A and B are finite we obtain the map $B \dashrightarrow K(X)$ since by Definition 9.8 the relative map $(V(R_0), K(X)) \to (R_0, K(X))$ is phantom. The map $X \dashrightarrow X$ is the identity map (formally it is the homotopy inverse of the composition $X \hookrightarrow K(X) \to L_ZK(X) \simeq X$). The composition

$$B \dashrightarrow K(X) \longrightarrow L_ZK(X) \simeq X \dashrightarrow X$$

is a trivialization of the composition $(B, A) \to (V(R_0), X) \to (R_0, X)$ which proves that the map $(V(R_0), X) \to (R_0, X)$ is phantom. □

Theorem 9.11. *Let Z be a fibrant simplicial set which is a retract of a simplicial compact space. Assuming Conjecture 8.3, any Z-local space X belongs to the smallest class $\mathfrak{L}(Z)$, of simplicial sets, which contains Z and is closed under weak equivalences and homotopy inverse limits.*

Proof. We construct a diagram

$$\begin{tikzcd}
& & & & & & X \arrow[dllllll] \arrow[dllll] \arrow[dll] \arrow[dl] \arrow[d] \\
K_{\omega+\omega} \arrow[d] & \cdots \arrow[l, hook'] & K_\omega \arrow[l, hook'] \arrow[d] & \cdots \arrow[l, hook'] & K_2 \arrow[l, hook'] \arrow[d] & K_1 \arrow[l, hook'] \arrow[d] & K_0 \arrow[l, hook'] \arrow[d] \\
V_{\omega+\omega} \arrow[r] & \cdots \arrow[r] & V_\omega \arrow[r] & \cdots \arrow[r] & V_2 \arrow[r] & V_1 \arrow[r] & V_0
\end{tikzcd}$$

in the following manner. Let $K_0 = K(X)$ and $V_0 = V(R_0)$. Let $K_{i+1} = K(K_i)$ and $V_{i+1} = V(V_i)$. For a limit ordinal i let $K_i = \text{hocolim}_{j<i} K_j$ and $V_i = \text{holim}_{j<i} V_j$. By Lemma 9.10 this diagram induces a relative tower

$$(V_{\omega+\omega}, X) \to ... \to (V_\omega, X) \to ... \to (V_2, X) \to (V_1, X) \to (V_0, X)$$

in which each map is phantom. Using Corollary 7.8 we see that the maps

$$(V_{\omega+\omega}, X) \to (V_\omega, X) \to (V_0, X)$$

are both skeleton-phantom. By Proposition 7.12 their composition is trivial. This shows that X is a homotopy retract of $V_{\omega+\omega}$. Since $V_{\omega+\omega}$ is in $\mathfrak{L}(Z)$ we see that X is in $\mathfrak{L}(Z)$. $\square$

As a corollary of Theorems 9.11 and 4.7 we obtain a result related to the work of Adámek and Rosický presented in [1]. In our case, however, the ability to interchange certain limit and colimit constructions comes as a conclusion, not an assumption.

Given a cardinal number κ we say that a colimit is κ-filtered, if it is indexed over a partially ordered set in which any subset of cardinality at most κ has an upper bound.

Corollary 9.12. *Let Z be a fibrant simplicial set which is a retract of a simplicial compact space. Assuming Conjecture 8.3 there exists a cardinal number κ such that the class $\mathfrak{L}(Z)$ is closed under κ-filtered homotopy colimits.*

Proof. By Theorem 9.11 the class $\mathcal{L}(Z)$ is the image of L_Z, the localization at Z. By Theorem 4.7 we have a map $f : A \to B$ such that $L_Z \simeq L_f$. This way $\mathcal{L}(Z)$ is the class of f-local simplicial sets. Let $\kappa = |A| + |B|$. By the definition (see (2.3)) and the small object argument the class of f-local simplicial sets is closed under κ-filtered homotopy colimits. □

The reader might consider this unexpected result as an evidence against Conjecture 8.3. We provide the following theorem which says that an analogue of this corollary is actually true in the category of groups.

Theorem 9.13. *Let G be a group which is a retract of a compact topological group. There is a cardinal number κ such that the least class of groups $\mathcal{L}(G)$ which contains G and is closed under limits is also closed under κ-filtered colimits.*

Proof. Consider localization at G in the category of groups. It is a coaugmented idempotent functor which takes to isomorphisms precisely those group homomorphisms $f : X \to Y$ for which

$$f^* : \mathrm{Hom}(Y, G) \to \mathrm{Hom}(X, G)$$

is a bijection. It was first observed by Pfenniger in [19] that such localization exists for any group G. More, it can be built by means of iterated inverse limits hence its image coincides with $\mathcal{L}(G)$.

On the other hand if G is a retract of a compact topological group we can retrace the arguments leading to Theorem 4.7 in the category of groups to see that in this case localization at G is an f-localization for some homomorphism $f : A \to B$. This way we see that $\mathcal{L}(G)$ which is equal to the image of L_f is closed under κ-filtered colimits for $\kappa = |A| + |B|$. □

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