An “almost” full embedding of the category of graphs into the category of groups

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The Functor
Construction of $F$
Applications
Summary

$F : \text{Graphs} \rightarrow \text{Groups}$

- full and faithful:

$$\text{Hom}(X, Y) \cong \text{Hom}(FX, FY)$$

- “almost” full:

$$\text{Hom}_{\text{Graphs}}(X, Y) \cup \{\ast\} \xrightarrow{\cong} \text{Rep}(FX, FY)$$

where $\text{Rep}(FX, FY) = \text{Hom}(FX, FY)/FY$
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\( F : \mathcal{G}raphs \rightarrow \mathcal{G}roups \)

Choice of categories: target

- \( \mathcal{G}roups \) - is interesting in itself
- \( \mathcal{G}roups \xrightarrow{B} \text{Ho} \) (unpointed homotopy category) yields, up to constant maps, a full embedding

\[ BF : \mathcal{G}raphs \rightarrow \text{Ho} \]
Choice of categories: source

\textit{Graphs} is very comprehensive and well researched. Many “non-homotopy” categories are contained in \textit{Graphs} as full subcategories:

- category of groups
- category of fields
- category of $R$-modules
- category of Hilbert spaces
- category of partially ordered sets
- category of simplicial sets
- category of metrizable spaces and continuous maps
- category of CW-complexes and continuous maps
- category of models of some first order theory
- many more

Tool: Adámek, Rosický \textit{Locally presentable and accessible categories}, Theorem 2.65.
Bass-Serre theory on groups acting on trees

\[ G = M \ast \left( N \ast P \right) \]

- If \( A \subseteq G \) is finite then it stabilizes a vertex of the tree hence is conjugated to a subgroup of \( M \) or \( P \).
- Take \( A = M \) finite, s.t. \( \text{Hom}(M, P) = * \) and \( M \rightarrow M \) is either trivial or an inner automorphism.
- Let \( N_M(N) = N, N \subseteq M \) does not extend to \( P \rightarrow M \), \( N \subseteq P \) extends uniquely to \( P \rightarrow P \) then \( M \subseteq G \) uniquely extends to \( G \rightarrow G \).
Bass-Serre theory on groups acting on trees

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- Take \( A = M \) finite, s.t. \( \text{Hom}(M, P) = \ast \) and \( M \rightarrow M \) is either trivial or an inner automorphism.
- Let \( N_M(N) = N \), \( N \subseteq M \) does not extend to \( P \rightarrow M \), \( N \subseteq P \) extends uniquely to \( P \rightarrow P \) then \( M \subseteq G \) uniquely extends to \( G \rightarrow G \).
If $A \subseteq G$ is finite then it stabilizes a vertex of the tree hence is conjugated to a subgroup of $M$ or $P$.

Take $A = M$ finite, s.t. $\text{Hom}(M, P) = \ast$ and $M \rightarrow M$ is either trivial or an inner automorphism.

Let $N_M(N) = N, N \subseteq M$ does not extend to $P \rightarrow M$, $N \subseteq P$ extends uniquely to $P \rightarrow P$ then $M \subseteq G$ uniquely extends to $G \rightarrow G$. 
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Construction of the functor $F$
Reduction to trees

\[ G = (M \ast N \ P) \ast N \ P \]

\[ \text{Rep}(G, G) = \{ \ast \} \cup \text{Hom}(2, 2) \]
Start with a graph $\Gamma$
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An “almost” full embedding ...
The Functor

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An “almost” full embedding...
Obtain graph of groups $G \Gamma$, define $F \Gamma = \text{colim } G \Gamma$
Example

\[ M_{23} \rightarrow N \oplus A(S_3 \oplus S_7) \rightarrow N \oplus A_11 \rightarrow N \oplus A(S_4 \oplus S_6) \rightarrow A_{11} \oplus A_{10} \rightarrow A_{11} \oplus A_{10} \rightarrow A_{11} \rightarrow \mathbb{Z}_{11} \times \mathbb{Z}_5 \]
Reduction to trees

\[ F \Gamma = F_1 \Gamma \ast F_0 \Gamma F_2 \Gamma \]
Two definitions of a localization $L : C \to C$

1. $L$ is a left adjoint of an inclusion $\mathcal{D} \subseteq C$ of some subcategory.

2. $L$ is a functor with coaugmentation $\eta : \text{Id} \to L$ such that $\eta_{LX} = L\eta_X : LX \to LLX$ is an isomorphism

Localizations may be viewed as projections onto the class of local objects $\mathcal{D}$ along the class of $L$-equivalences $\mathcal{E} = \{f \mid Lf$ is an equivalence$\}$

For every $f : A \to B$ in $\mathcal{E}$, an $L$-equivalence and $Z$ in $\mathcal{D}$, an $L$-local object we have:

$$\text{Hom}(B, Z) \xrightarrow{\sim} \text{Hom}(A, Z)$$

$$\text{map}(B, Z) \xrightarrow{\sim} \text{map}(A, Z)$$
**Orthogonality classes**

If for $f : A \to B$ and $Z$ we have

$$\text{Hom}(B, Z) \xrightarrow{\cong} \text{Hom}(A, Z)$$

$$\text{map}(B, Z) \xrightarrow{\cong} \text{map}(A, Z)$$

then we say that $f$ is *orthogonal* to $Z$ and write $f \perp Z$.

A pair $(\mathcal{E}, \mathcal{D})$ is orthogonal if $\mathcal{E} = \mathcal{D}^{\perp}$ and $\mathcal{D} = \mathcal{E}^{\perp}$.

A localization always yields an orthogonal pair.

Whether every orthogonal pair yields a localization depends on set theory.

*in Cahiers:*

- NO is consistent with ZFC
- weak Vopěnka’s principle is equivalent to YES
Orthogonality classes

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in Graphs:

- NO is consistent with ZFC
- weak Vopěnka’s principle is equivalent to YES
More properties of $F : \mathbf{Graphs} \to \mathbf{Groups}$

1. $\text{Hom}_{\mathbf{Graphs}}(X, Y) \cup \{\ast\} \cong \text{Rep}(FX, FY)$
2. $f \perp Z$ if and only if $Ff \perp FZ$
3. $F$ preserves directed colimits
4. $F$ preserves intersections and countably co-directed limits
5. $\Delta \subseteq \Gamma$ implies $F\Delta \subseteq F\Gamma$
6. for every $g \in F\Gamma$ there exists a finite subgraph $\Delta \subseteq \Gamma$ s.t. $g \in F\Delta$
7. $F$ does not preserve products.
Large localizations of finite groups

Theorem
There exist localizations $L : \text{Groups} \rightarrow \text{Groups}$ whose values $LM$ on a finite group $M$ have arbitrarily large cardinalities.

Proof.
Vopěnka (1965): there exist arbitrarily large graphs $\Gamma$ s.t. $\text{Hom}(\Gamma, \Gamma) = \{id\}$.
The inclusion $i : \emptyset \subseteq \Gamma$ is orthogonal to $\Gamma$.
$Fi \perp F\Gamma$
If $L = L_{Fi}$ then $LM = F\Gamma$.

Theorem

The following are equivalent:

1. Every orthogonal pair \((\mathcal{E}, \mathcal{D})\) in \(\text{Groups}\) is associated with a localization.

2. Every orthogonal pair \((\mathcal{E}, \mathcal{D})\) in \(\text{Graphs}\) is associated with a localization (weak Vopěnka’s principle).

\[ 2 \implies 1 \text{ was proved by Adámek and Rosický (1994)} \]
\[ 1 \implies 2 \text{ follows from properties of } F \]
Theorem

The following are equivalent:

1. For every orthogonal pair \((\mathcal{E}, \mathcal{D})\) in Groups there exists a homomorphism \(f\) such that \(\mathcal{D} = \{f\}^\perp\).

2. For every orthogonal pair \((\mathcal{E}, \mathcal{D})\) in Graphs there exists a map \(f\) such that \(\mathcal{D} = \{f\}^\perp\) (Vopěnka’s principle).

3. For every orthogonal pair \((\mathcal{E}, \mathcal{D})\) in \(Ho\) there exists a map \(f\) such that \(\mathcal{D} = \{f\}^\perp\).

2 \implies 3 was proved by Casacuberta, Scevenels, Smith (2005)
1 \implies 2 and 3 \implies 1 follow from properties of \(F\)
Summary

An almost full embedding $F : \text{Graphs} \rightarrow \text{Groups}$

$$\text{Hom}_{\text{Graphs}}(X, Y) \cup \{\ast\} \xrightarrow{\cong} \text{Rep}(FX, FY)$$

is a “black box” tool translating some categorical constructions from many point-set categories to the category of groups (or to the homotopy category).

Question
Is there an embedding $F : \text{Graphs} \rightarrow \text{Ab} – \text{Groups}$ such that $f \perp Z$ if and only if $Ff \perp FZ$. 
An almost full embedding $F : \text{Graphs} \to \text{Groups}$

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Is there an embedding $F : \text{Graphs} \to \text{Ab} - \text{Groups}$ such that $f \perp Z$ if and only if $Ff \perp FZ$. 

Thank you