THE HAAR MEASURE PROBLEM

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Abstract. An old problem asks whether every compact group has a Haar-nonmeasurable subgroup. A series of earlier results reduce the problem to infinite metrizable profinite groups. We provide a positive answer, assuming a weak, potentially provable, consequence of the Continuum Hypothesis. We also establish the dual, Baire category analogue of this result.

1. Introduction

Every infinite compact group has a unique translation-invariant probability measure, its *Haar measure*. Vitali sets (complete sets of coset representatives) with respect to a countably infinite subgroup show that such groups have nonmeasurable subsets. We consider the following old problem.

Haar Measure Problem. *Does every infinite compact group have a nonmeasurable subgroup?*

The Haar Measure Problem dates back at least to 1963, when Hewitt and Ross gave a positive answer for abelian groups [5, Section 16.13(d)]. It was explicitly formulated in a paper of Saeki and Stromberg [8]. The problem remains open despite substantial efforts [4, 3, and references therein].

Hernández, Hofmann, and Morris proved that if all subgroups of an infinite compact group are measurable, then the group must be profinite and metrizable [4, Theorem 2.3 and Corollary 3.3]. Building on that, Brian and Mislove proved that a positive answer to the Haar Measure Problem is *consistent* (relative to the usual axioms of set theory) [3, Theorem 2.5]. We repeat their argument, for its elegant simplicity, and since this result will take care of the easier case of our main theorem: Let G be an infinite, metrizable profinite group. As a measure space, the group G is isomorphic to the Cantor space with the Lebesgue measure. Let $\mathfrak{c}$ denote the cardinality of the continuum. Consistently, there is in the Cantor space, and thus in G, a nonnull set A of cardinality smaller than $\mathfrak{c}$. The subgroup of G generated

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by A is nonnull, and its cardinality is smaller than $\mathfrak{c}$. Since sets of positive measure have cardinality $\mathfrak{c}$, the group generated by A is nonmeasurable.

Brian and Mislove's observation can be viewed as a solution of the Haar Measure Problem under the hypothesis that there is a nonnull set of cardinality smaller than $\mathfrak{c}$. This hypothesis violates the Continuum Hypothesis. We will show that the Continuum Hypothesis also implies a positive solution. Moreover, for our proof we only assume a weak consequence of the Continuum Hypothesis, which is provable for some groups, and may turn out provable for all groups. A proof of our hypothesis, if found, would settle the Haar Measure Problem.

2. The main theorem

Throughout this section, we fix an arbitrary infinite metrizable profinite group G, and let μ be its Haar probability measure. For each natural number n, the Haar probability measure on the group G^n is the product measure, which is also denoted μ.

Let H be a subgroup of G, and $X = \{x_1, x_2, \dots\}$ be a countable set of variables. The set of all words in the alphabet $H \cup \{x_1^{\pm 1}, x_2^{\pm 1}, \dots\}$ is denoted $H[X]$. Each word $w \in H[X]$ depends on finitely many parameters from the subgroup H, and finitely many variables; let $|w|$ denote the number of variables in w. We view the word w as a continuous function from $G^{|w|}$ to G defined by substituting the group elements for the variables.

Definition 1. Let e be the identity element of the group G. A *Markov set* is a set of the form $w^{-1}(e)$ for $w \in G[X] \setminus G$.

Lemma 2. *For each element $b \in G$, and each word $w \in G[X] \setminus G$, the set $w^{-1}(b)$ is Markov.*

Proof. Consider the word wb^{-1}. □

For a natural number $n \geq 2$, a set $A \subseteq G^n$, and an element $g \in G$, we define

$$A_g := \{\, h \in G^{n-1} : (h, g) \in A \,\},$$

the fiber of the set A over the point g in the group G^{n-1}.

The Markov sets were studied by Markov, as the sets that are closed in all group topologies on G. This explains our terminology. Note that, in particular, Markov sets are closed (and thus measurable), and so are their fibers.

Definition 3. A *Markov null set* is a Markov subset of some finite power of the group G, that is also null with respect to the Haar measure μ. A set N is *Fubini–Markov* if either of the following two cases holds:

(1) The set N is a Markov null subset of G.

(2) There are a natural number $n \geq 2$ and a Markov null set $A \subseteq G^n$ such that $N = \{ g \in G : \mu(A_g) > 0 \}$.

While Markov sets may be subsets of an arbitrary power of G, Fubini–Markov sets are always subsets of G. By the Fubini Theorem, we have the following observation.

Lemma 4. *Every Fubini–Markov set is null.* □

We define a cardinal invariant of the group G.

Definition 5. The *Fubini–Markov number* of G, denoted $\mathfrak{fm}(G)$, is the minimal number of Fubini–Markov sets in G whose union has full measure.

Since a countable union of null sets is null, the Fubini–Markov number of a group is necessarily uncountable.

Example 6. For the Cantor group, we have $\mathfrak{fm}(\mathbb{Z}_2^{\mathbb{N}}) = \mathfrak{c}$. Indeed, let G be an *abelian* infinite metrizable profinite group, and N be a Fubini–Markov set. We consider the two cases in the definition.

Assume that $N = w^{-1}(0) \subseteq G$ for some one-variable word $w \in G[X] \setminus G$. Since the group G is abelian, we have $w(x) = x + a$ for some $a \in G$. Then $w(x) = 0$ if and only if $x = -a$, and thus N is a singleton.

Next, for a natural number $n \geq 2$, let $w^{-1}(0) \subseteq G^n$ be a Markov null set, where $w \in G[X] \setminus G$ and $|w| = n$. Since the group G is abelian, we have $w := x_1 + \cdots + x_n + a$, for some $a \in G$. Then, for each $g \in G$, we have

$$(w^{-1}(0))_g = \{ (h_1, \ldots, h_{n-1}) : h_1 + \cdots + h_{n-1} + g + a = 0 \},$$

and thus $(w^{-1}(0))_g$ is a Lipschitz image of the null set $G^{n-2} \times \{0\}$. It follows that $\mu((w^{-1}(0))_g) = 0$ for all $g \in G$, and we have, in the definition, $N = \emptyset$.

A union of fewer than $\mathfrak{c}$ sets that are at most singletons cannot cover a full measure set.

We arrive at our main theorem. Let $\mathcal{N}$ be the ideal of null sets in the Cantor space, and $\mathrm{non}(\mathcal{N})$ be the minimal cardinality of a nonnull subset of the Cantor space. We settle the Haar Measure Problem for groups G with $\mathrm{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$. By Lemma 4, the Continuum Hypothesis implies $\mathrm{non}(\mathcal{N}) = \mathfrak{fm}(G)$. Example 6 shows that for some groups our hypothesis is provable. We do not know whether it is provable for all infinite metrizable profinite groups G. The numbers $\mathfrak{fm}(G)$ are provably larger than some classical cardinal invariants of the continuum; we will return to this in Section 3.

Theorem 7. *Let G be an infinite metrizable profinite group with* $\mathrm{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$*. Then G has a Haar-nonmeasurable subgroup.*

Proof. If $\mathrm{non}(\mathcal{N}) < \mathfrak{c}$, then the Brian–Mislove argument applies, namely, every nonnull set of cardinality $\mathrm{non}(\mathcal{N})$ generates a nonmeasurable subgroup of G; see Section 1 for the details. We may thus assume that $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$. By our hypothesis, we have $\mathfrak{fm}(G) = \mathfrak{c}$.

Let $\{ N_\alpha : \alpha < \mathfrak{c} \}$ be the family of G_δ null sets. Every null set is contained in some N_α. We define a transfinite, increasing chain of subgroups H_α of G for $\alpha < \mathfrak{c}$. Let H_0 be a countable dense subgroup of G. Let $w \in H_0[X] \setminus H_0$. For distinct elements $b_1, b_2 \in G$, the sets $w^{-1}(b_1)$ and $w^{-1}(b_2)$ are disjoint. Since Markov sets are closed (and thus measurable), the set

$$P_w := \{ b \in G : \mu(w^{-1}(b)) > 0 \}$$

is countable. Since the set $H_0[X] \setminus H_0$ is countable, there is an element

$$b \in G \setminus \Big(H_0 \cup \bigcup_{w \in H_0[X] \setminus H_0} P_w \Big).$$

This element b will remain outside our subgroups throughout the construction.

We proceed by induction. For a limit ordinal α, we set $H_\alpha := \bigcup_{\beta<\alpha} H_\beta$. For a successor ordinal $\alpha = \beta + 1 < \mathfrak{c}$, we assume, inductively, that $|H_\beta| < \mathfrak{c}$, $b \notin H_\beta$, and the sets $w^{-1}(b)$ are null for all words $w \in H_\beta[X] \setminus H_\beta$.

Since $|H_\beta[X]| < \mathfrak{c}$, the set

$$S := \bigcup_{\substack{w \in H_\beta[X] \\ |w|=1}} w^{-1}(b) \cup \bigcup_{\substack{w \in H_\beta[X] \\ |w|\geq 2}} \{ g \in G : \mu\big((w^{-1}(b))_g\big) > 0 \}.$$

is a union of fewer than $\mathfrak{fm}(G)$ Fubini–Markov sets, and thus does not have full measure. Pick an element $g_\alpha \in G \setminus (S \cup N_\alpha)$. Let $H_\alpha := \langle H_\beta, g_\alpha \rangle$. We verify that the inductive hypotheses are preserved.

Fix $c \in H_\alpha$. There is a word $w \in H_\beta[X]$ with $|w| = 1$ such that $w(g_\alpha) = c$, and thus $g_\alpha \in w^{-1}(c)$. Since $g_\alpha \notin S \supseteq w^{-1}(b)$, we have $c \neq b$. This shows that $b \notin H_\alpha$. Next, consider an arbitrary word $v = v(x_1, \dots, x_n) \in H_\alpha[X] \setminus H_\alpha$. There is a word $w = w(x_1, \dots, x_n, x_{n+1}) \in H_\beta[X] \setminus H_\beta$ such that

$$v(x_1, \dots, x_n) = w(x_1, \dots, x_n, g_\alpha).$$

Since $g_\alpha \notin S$ and $|w| \geq 2$, the set $v^{-1}(b) = (w^{-1}(b))_{g_\alpha}$ is null.

Having defined all subgroups H_α for $\alpha < \mathfrak{c}$, let $H := \bigcup_{\alpha<\mathfrak{c}} H_\alpha$. Then H is a proper (since $b \notin H$), dense (since $H_0 \subseteq H$), nonnull (since $H \nsubseteq N_\alpha$ for all α) subgroup of G. Assume that H is measurable. Then it has positive measure, and by the Steinhaus Theorem, it contains an open set. Since it is dense, we have $H = G$, a contradiction. □

Our main theorem also has a dual, Baire category version. Let $\mathcal{M}$ be the ideal of meager (Baire first category) subsets of the Cantor space. We define *Kuratowski–Ulam–Markov sets* by changing *null* to *meager* in Definition 3. In this case, item (2) of the definition becomes

$$N = \{ g \in G : A_g \text{ is nonmeager} \}.$$

By the Kuratowski–Ulam Theorem, Kuratowski–Ulam–Markov sets are meager. Similarly, we dualize Definition 5 to define the Kuratowski–Ulam–Markov number $\mathfrak{kum}(G)$. Let $\text{non}(\mathcal{M})$ be the minimal cardinality of a nonmeager subset of the Cantor space.

Theorem 8. *Let G be an infinite metrizable profinite group with* $\text{non}(\mathcal{M}) \leq \mathfrak{kum}(G)$*. Then G has a subgroup that does not have the property of Baire.*

Proof. If $\text{non}(\mathcal{M}) < \mathfrak{c}$, then any nonmeager set of cardinality $\text{non}(\mathcal{M})$ generates a nonmeager subgroup of G that does not have the Baire property (nonmeager sets with the Baire property have cardinality $\mathfrak{c}$) [3, Theorem 2.5].

Thus, assume that $\text{non}(\mathcal{M}) = \mathfrak{c}$. We proceed as in the proof of Theorem 7, replacing *null* by *meager* and sets of positive measure by *nonmeager sets* (the relevant sets are closed). For the choice of the element b, we observe that closed nonmeager sets are, in particular, not nowhere dense, and thus have nonempty interior. Our group G is homeomorphic to the Cantor space, and thus there are in G at most countably many disjoint open sets.

We thus obtain a proper dense nonmeager subgroup of G. To conclude the proof, we use the Pettis Theorem [6, Theorem 9.9], the category-theoretic dual of the Steinhaus Theorem: If a set $H \subseteq G$ is nonmeager and has the Baire property, then the quotient $H^{-1}H$ has nonempty interior. □

3. Bounds on the Fubini–Markov number

Here too, all groups are assumed to be infinite metrizable profinite. Theorem 7 applies to groups G with $\text{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$. We saw that abelian groups have $\mathfrak{fm}(G) = \mathfrak{c}$, but the following conjecture remains open.

Conjecture 9. *For each infinite metrizable profinite group G, we have* $\text{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$*.*

A proof of this conjecture would settle the Haar Measure Problem, but it may turn out unprovable (and thus undecidable). In this case, well-studied lower bounds on $\mathfrak{fm}(G)$ are useful.

The *covering number* of an ideal $\mathcal{I}$ of subsets of the Cantor space, denoted $\text{cov}(\mathcal{I})$, is the minimal number of elements of $\mathcal{I}$ needed to cover the Cantor space. Since Fubini–Markov sets are null, we have $\text{cov}(\mathcal{N}) \leq \mathfrak{fm}(G)$ for all groups G: A set of full measure needs just one additional null set to cover the entire space. The following result provides a tighter estimate, in the sense that it is provably larger, and consistently strictly larger.

Let $\mathcal{E}$ be the σ-ideal generated by the closed null sets in the Cantor space. The following proof establishes, in particular, that the family of Fubini–Markov sets is contained in $\mathcal{E}$.

Proposition 10. *For each infinite metrizable profinite group G, we have* $\operatorname{cov}(\mathcal{E}) \leq \mathfrak{fm}(G)$.

Proof. Brian proved that $\operatorname{cov}(\mathcal{E})$ is equal to the minimal number of closed null subsets of the Cantor space that cover a set of positive measure [2]. Thus, it suffices to prove that every Fubini–Markov subset N of G is a countable union of closed null subsets of G.

Let N be a Fubini–Markov subset of G. If N is Markov null, then it is closed and null. It remains to consider the case that

$$N = \{\, g \in G : \mu\big((w^{-1}(e))_g\big) > 0 \,\},$$

where $w \in G[X]$ has $|w| \geq 2$.

For each natural number k, the subset

$$N_k := \{\, g \in G : \mu\big((w^{-1}(e))_g\big) \geq 1/k \,\}$$

of N is null (Lemma 4), and $N = \bigcup_k N_k$. Each set N_k is closed: Let $g \in G \setminus N_k$. There is an open set V in $G^{|w|-1}$ such that $(w^{-1}(e))_g \subseteq V$ and $\mu(V) < 1/k$. Let P be the projection of the compact set $w^{-1}(e) \setminus (V \times G)$ on the last coordinate. The set $G \setminus P$ is an open neighborhood of g in G. For each element $h \in G \setminus P$, we have $(w^{-1}(e))_h \subseteq V$. Thus, $\mu((w^{-1}(e))_h) \leq \mu(V) < 1/k$, and $(G \setminus P) \cap N_k = \emptyset$. □

Corollary 11. *Assume that* $\operatorname{non}(\mathcal{N}) \leq \operatorname{cov}(\mathcal{E})$*. Then every infinite compact group has a nonmeasurable subgroup.*

Proof. Theorem 7 and Proposition 10. □

Since $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$, we have

$$\max\{\operatorname{cov}(\mathcal{M}), \operatorname{cov}(\mathcal{N})\} \leq \operatorname{cov}(\mathcal{E}).$$

It follows that if $\operatorname{non}(\mathcal{N}) \leq \max\{\operatorname{cov}(\mathcal{M}), \operatorname{cov}(\mathcal{N})\}$, then every compact group has a nonmeasurable subgroup. The hypothesis $\operatorname{non}(\mathcal{N}) \leq \operatorname{cov}(\mathcal{E})$ is not provable; this follows from known upper bounds on $\operatorname{cov}(\mathcal{E})$ [7].

We conclude this section with a simple sufficient condition for our main theorem. This condition is stronger than the hypothesis $\operatorname{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$, but it may still be provable.

Definition 12. Let G be an infinite metrizable profinite group. For a natural number n, let κ_n be the minimal number of Markov null subsets of the group G^n whose union is not null. The *Markov number* of G is the cardinal number $\mathfrak{mar}(G) := \min_n \kappa_n$.

Problem 13. *In Definition 12, is the sequence $\kappa_1, \kappa_2, \dots$ constant? In particular, is it provable that $\mathfrak{mar}(G)$ is equal to the minimal number of Markov null subsets of the group G whose union is not null?*

Lemma 14. *Let G be an infinite metrizable profinite group. Then:*

(1) $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{mar}(G) \leq \mathrm{non}(\mathcal{N})$,

(2) $\mathfrak{mar}(G) \leq \mathfrak{fm}(G)$.

Proof. (1) Markov sets are closed and null. It follows that the minimal number of closed null sets in the Cantor space whose union is not null is at most $\mathfrak{mar}(G)$. The former number is equal to $\mathrm{cov}(\mathcal{M})$ [1, Theorem 2.6.14]. Since every singleton is a Markov set (consider the word $w(x) = x$), we have $\mathfrak{mar}(G) \leq \mathrm{non}(\mathcal{N})$.

(2) Let $\mathcal{F}$ be a family of Fubini–Markov subsets of the group G with $|\mathcal{F}| < \mathfrak{mar}(G)$. For each element of $\mathcal{F}$, fix a Markov null set witnessing its being Fubini–Markov, and let $\mathcal{A}$ be the family of these Markov sets. For a natural number n, let $A_n := \bigcup \{ A \in \mathcal{A} : A \subseteq G^n \}$. Since $|\mathcal{A}| < \mathfrak{mar}(G)$, the set A_n is null. By the Fubini Theorem, the set

$$S := A_1 \cup \bigcup_{n=2}^{\infty} \{ g \in G : \left(A_n\right)_g \text{ is not null} \}$$

is null. Then S is null, and $\bigcup \mathcal{F} \subseteq S$. □

Thus, the following conjecture implies a positive solution to the Haar Measure Problem.

Conjecture 15. *For each infinite metrizable profinite group G, we have $\mathrm{non}(\mathcal{N}) = \mathfrak{mar}(G)$.*

Conjecture 15 holds when restricted to abelian groups, since Lipschitz images of sets of the form $A \times \{0\}$, with $|A| < \mathrm{non}(\mathcal{N})$, are null (see Example 6)

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