Homotopical localizations at a space ⋆

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Abstract

Our main motivation for the work presented in this paper is to construct a localization functor, in a certain sense dual to the f-localization of Bousfield and Farjoun, and to study some of its properties. We succeed in a case which is related to the Sullivan profinite completion. As a corollary we prove the existence of certain cohomological localizations.

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1 Introduction

We can view f-localization as the initial coaugmented idempotent functor on the homotopy category which takes a map f to an equivalence. In [1] Bousfield used the small object argument to prove that f-localizations exist for all maps f. The role of these functors was especially exposed in 1990's when they were put in a convenient framework in terms of mapping complexes. A survey of related methods can be found in [12] and [7]. It seems natural to ask if a dual notion of a localization at a space Z, that is the terminal idempotent functor with a given space Z in its image (Definition 5), might not also be interesting. The main reason these localizations have not been considered very

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much is that they are not known to exist in general, even in the stable case (see Chapter 7 in [14]).

As every homological localization can be realized as an f-localization, every cohomological localization, provided it exists, is a localization at a suitable space. Research towards establishing the existence of cohomological localizations was briefly summarized in 2.6 of [6].

Here we prove the existence of localizations at compactly topologized spaces (Definition 11 and Theorem 19). Examples of such spaces include the ones which are profinite completions of another space, mapping complexes with a profinitely completed target, and others. This result allows us to construct an idempotent approximation to the Sullivan profinite completion (Theorem 23).

We would like to be able to prove the existence of localization at an arbitrary space without relying on the compactness condition, and there is some evidence that such localizations should exist at abelian Eilenberg-Mac Lane spaces. These would form "truncated localizations at an ordinary cohomology theory", an analogue of "truncated localizations at a homology theory" whose existence was shown by Ohkawa in [16]. It would also be interesting to find how such localizations act on spaces and how they are related to those f-localizations, that do not correspond to a localization at any space.

Casacuberta, Scevenels and Smith investigated in [9] dependence on certain large cardinal axioms of a more general question, from a positive answer to which the existence of localizations at any space would follow. Despite extensive efforts we were unable to avoid similar set theoretic problems in our attempts to prove the existence of localizations at a general space, nor were we able to disprove it under some large cardinal axioms.

The main Theorem 19 is proved in section 5. In section 6 we describe an idempotent approximation to the Sullivan profinite completion and prove the existence of certain cohomological localizations.

The paper is written simplicially. We use terms "space" and "simplicial set" as synonyms choosing the second one wherever confusion with compact topological space might occur or to emphasize it when we work on the point set level rather than in the homotopy category. To make the presentation more accessible, we frequently work in the pointed homotopy category Ho_*. Adjective "compact" always means "compact Hausdorff".

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2 Localizations

In this section we collect basic definitions and facts related to homotopical localizations.

A functor L is called *coaugmented* if it comes with a natural transformation $\eta_X : X \to LX$ from the identity to L. A coaugmented functor is *idempotent* if in the diagram

$$\begin{array}{ccc} X & \longrightarrow & LX \\ \downarrow & & \downarrow \eta_{LX} \\ LX & \xrightarrow{L\eta_X} & LLX \end{array}$$

the maps η_{LX} and $L\eta_X$ are equivalences and $\eta_{LX} = L\eta_X$.

Definition 1 *A coaugmented idempotent functor is called a* localization.

Although this definition makes sense in any category we will consider only localizations in the homotopy category Ho_* of pointed simplicial sets (spaces). A space Z is said to be L*-local* if the map $\eta_Z : Z \to LZ$ is an equivalence. It is straightforward to check that the class of L-local spaces uniquely determines and is determined by the functor L. A map $g : X \to Y$ is an L*-equivalence* if Lg is an equivalence. There is a natural ordering of localizations as described below.

Definition 2 *Given two localization functors L_1 and L_2 we say that $L_1 \leq L_2$ if one of the equivalent conditions hold:*

(i) there is a natural transformation $L_1 \to L_2$ giving $L_2 L_1 \simeq L_2$
(ii) any L_1-equivalence is also an L_2-equivalence
(iii) any L_2-local space is also L_1-local

This definition is an obvious extension of the ordering in the Bousfield lattice of f-localizations (4.3 in [6]).

Given a map $f : A \to B$ we say that a fibrant space Z is f*-local* if the induced map of function complexes

$$f^* : \mathrm{map}_*(B, Z) \to \mathrm{map}_*(A, Z) \tag{3}$$

is an equivalence. If Z is connected the condition above is equivalent to the one that the induced map of unbased function complexes

$$f^* : \mathrm{map}(B, Z) \to \mathrm{map}(A, Z)$$

is an equivalence.

A map $g : X \to Y$ is an *f-equivalence* if any f-local space is also g-local. This means that for any fibrant space Z if

$$f^* : \mathrm{map}_*(B, Z) \xrightarrow{\simeq} \mathrm{map}_*(A, Z)$$

then

$$g^* : \mathrm{map}_*(Y, Z) \xrightarrow{\simeq} \mathrm{map}_*(X, Z).$$

Definition 4 *An* f-localization *is a localization functor L_f such that the following conditions hold:*

(i) The classes of f-equivalences and L_f-equivalences coincide.
(ii) The classes of f-local and L_f-local spaces coincide.
(iii) The map $X \to L_f X$ is an f-equivalence and $L_f X$ is f-local.
(iv) L_f is the initial localization functor such that the map f is an L_f-equivalence.

For a map f, there are obvious implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$.

The existence of f-localizations for arbitrary maps f was proved by Bousfield [1] and Farjoun [11].

Let Z be a fibrant space. We say that a map $g : X \to Y$ is a *Z-equivalence* if the induced map of function complexes

$$g^* : \mathrm{map}_*(Y, Z) \to \mathrm{map}_*(X, Z)$$

is an equivalence. A fibrant space K is *Z-local* if it is g-local for all Z-equivalences g. This means that for any g if

$$g^* : \mathrm{map}_*(Y, Z) \xrightarrow{\simeq} \mathrm{map}_*(X, Z)$$

then

$$g^* : \mathrm{map}_*(Y, K) \xrightarrow{\simeq} \mathrm{map}_*(X, K)$$

Definition 5 *A* localization at Z *is a localization functor L_Z such that the following conditions hold:*

(i) the classes of Z-equivalences and L_Z-equivalences coincide.
(ii) the classes of Z-local and L_Z-local spaces coincide.
(iii) The map $X \to L_Z X$ is a Z-equivalence and $L_Z X$ is Z-local.
(iv) L_Z is the terminal localization functor such that the space Z is L_Z-local.

For a space Z, there are obvious implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$.

The implication $(iv) \Rightarrow (iii)$ is obvious when L_Z in the sense of *(i) - (iii)* exists. The only problem might arise if L_Z exists in the sense of *(iv)* but not *(i) - (iii)*, that is, a terminal localization T such that Z is T-local exists but not all T-local spaces are Z-local (condition *(ii)*). Suppose K is such a T-local but not Z-local space. Then there is a Z-equivalence $f : A \to B$ which is not a K-equivalence. Thus K is T-local but not f-local hence L_f is not less than T which contradicts *(iv)*.

The existence of localization at a given space Z is not known in general.

It is clear that the classes of Z-equivalences and f-equivalences are closed under arbitrary homotopy colimits. Also the classes of Z-local and f-local spaces are closed under arbitrary homotopy limits.

Lemma 6 *Suppose that for a certain space Z there is a set of Z-equivalences $\{f_\alpha\}$ such that every Z-equivalence can be presented as a homotopy colimit of elements of the set $\{f_\alpha\}$. Then the localization at Z is simply an f-localization for $f = \bigvee f_\alpha$.*

3 A characterization of Z-equivalences

In this section we recall Lemma 8. Although it is not new we prove it here since we didn't find an appropriate reference.

We say that a map $f : A \to B$ has a left lifting property (LLP) with respect to a map $g : C \to D$ if any diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ B & \longrightarrow & D \end{array}$$

admits the dashed map. For the sake of clarity we will use the term homotopy LLP when the lift we have in mind is in the homotopy category.

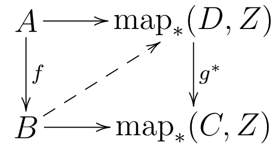
Lemma 7 *Let $f : A \to B$ and $g : C \to D$ be maps in Ho_*. The map f has the homotopy LLP with respect to*

$$g^* : map_*(D, Z) \to map_*(C, Z)$$

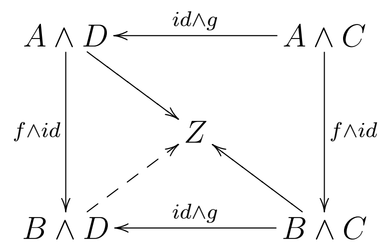
if and only if g has the homotopy LLP with respect to

$$f^* : map_*(B, Z) \to map_*(A, Z)$$

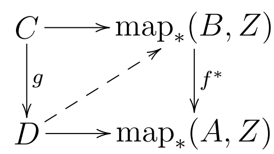
PROOF. We use adjointness to note that the existence of a dashed lift in the diagram



is equivalent to the existence of the dashed map in the following diagram.



This in turn is equivalent to the lifting property as indicated on the next diagram.



□

Lemma 8 *Let $g : \bigvee_{n\geq 0} S^n \to \bigvee_{n\geq 0} S^n$ be the trivial map. A map $f : A \to B$ is a Z-equivalence if and only if it has the homotopy LLP with respect to*

$$g_+^* : map_*((\bigvee_{n\geq 0} S^n)_+, Z) \to map_*((\bigvee_{n\geq 0} S^n)_+, Z)$$

PROOF. By Lemma 7 f has the homotopy LLP with respect to g_+^* if and only if g_+ has the homotopy LLP with respect to $f^* : \mathrm{map}_*(B, Z) \to \mathrm{map}_*(A, Z)$. Obviously if f^* is a weak equivalence then g_+ has the homotopy LLP hence the proof will be complete once we show that the homotopy LLP for g_+ implies that f^* is a weak equivalence. We see that if g_+ has the homotopy LLP with respect to f^* then all the maps $g_+^n : S_+^n \to \{*\}_+ \to S_+^n$ for $n \geq 0$ have the homotopy LLP. The case $n = 0$ implies that f^* induces a bijection on the components.

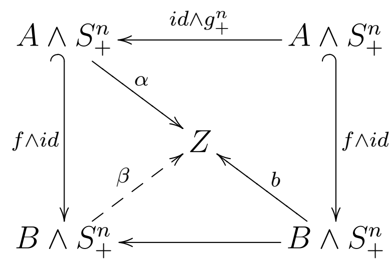
We are proving that f^* induces isomorphisms of homotopy groups of the corresponding components. Assume that f is an inclusion $A \hookrightarrow B$ of simplicial sets and Z is a fibrant simplicial set. We fix any map $b_0 : B \to Z$ as a basepoint of $\mathrm{map}_*(B, Z)$ and $a_0 = f^*(b_0)$ as a basepoint of $\mathrm{map}_*(A, Z)$. The homotopy LLP for g_+^n for $n > 0$ implies that f^* induces bijections of the homotopy groups modulo the action of the fundamental group:

$$\pi_n(\mathrm{map}_*(B, Z), b_0)/\sim \; \to \pi_n(\mathrm{map}_*(A, Z), a_0)/\sim$$

Since 0 is fixed by the action of the fundamental group we see that

$$f_n^* : \pi_n(\mathrm{map}_*(B,Z), b_0) \hookrightarrow \pi_n(\mathrm{map}_*(A,Z), a_0)$$

is a monomorphism for $n > 0$. Choose an element $\tilde{\alpha} \in \pi_n(\mathrm{map}_*(A,Z), a_0)$. It is represented by some $\alpha : A \wedge S^n_+ \to Z$ such that $\alpha|_{A = A\wedge\{*\}_+} = a_0$. We construct the following diagram.

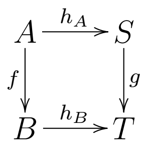


The map b is the composition $B \wedge S^n_+ \to B \wedge \{*\}_+ = B \xrightarrow{b_0} Z$. The diagram commutes by the definition of a_0 as $b_0 f$. By the proof of Lemma 7 the assumption that g^n_+ has the homotopy LLP with respect to f^* implies the existence of the dashed map β which closes this diagram up to homotopy. Since f^* is a bijection on components we see that $\beta|_{B\wedge\{*\}_+} : B \to Z$ must be homotopic to b_0. Since $A \wedge \{*\}_+ \hookrightarrow B \wedge S^n_+$ is a cofibration we can find β_1, homotopic to β, such that $\beta_1|_{B\wedge\{*\}_+} = b_0$. We see that β_1 induces an element $\tilde{\beta}$ in $\pi_n(\mathrm{map}_*(B,Z), b_0)$ such that $f^*(\tilde{\beta}) = \tilde{\alpha}$ hence f^* is a weak equivalence. □

4 Categories of pairs and topologized objects

In this section we collect some categorical definitions and facts which will be used in section 5. Some statements refer to a general category $\mathcal{C}$, however for us the interesting cases are when $\mathcal{C} = \mathcal{S}_*$ (pointed simplicial sets) or $\mathcal{C} = Ho_*$.

Definition 9 *Given a category $\mathcal{C}$ we will denote by $\mathcal{C}^2$ the usual* category of pairs *whose objects are the maps in $\mathcal{C}$ and whose maps are commutative squares in $\mathcal{C}$ as below.*



Following Bousfield and Friedlander (see [2] A3) we introduce a model category structure on $\mathcal{C}^2$.

Definition 10 *Let $\mathcal{C}$ be a model category. A map $h : f \to g$ as in Definition 9 is called a* weak equivalence *(respectively* fibration*) if both h_A and h_B are* *weak equivalences (respectively fibrations). It is a* cofibration *if* $h_A : A \to S$ *and* $(h_B, g) : B \amalg_A S \to T$ *are cofibrations. This implies that* $h_B : B \to T$ *is also a cofibration.*

Note that an object $f : A \to B$ is cofibrant in $\mathcal{C}^2$ if A is cofibrant in $\mathcal{C}$ and the map f is a cofibration in $\mathcal{C}$. It is fibrant if both S and T are fibrant in $\mathcal{C}$.

We will be interested in $ho\mathcal{S}^2_*$ the homotopy category of pairs when $\mathcal{C} = \mathcal{S}_*$ the category of pointed simplicial sets. The obvious functor $F : ho\mathcal{S}^2_* \to Ho^2_*$ induces equivalence of categories.

Some of the definitions below are chosen after [10]. For any category $\mathcal{C}$ and an object X of $\mathcal{C}$ a *topologized object over* X is a factorization

$$\begin{array}{ccccc} & & \mathrm{Top} & & \\ & \overset{X^{\#}}{\nearrow} & & \overset{G}{\searrow} & \\ \mathcal{C}^{op} & & \xrightarrow[\mathcal{C}(-,X)]{} & & \mathrm{Sets} \end{array}$$

where G is the forgetful functor. We say that a morphism $f : X \to Y$ is *continuous* if it induces a natural transformation $f^{\#} : X^{\#} \to Y^{\#}$, that is to say, the map $\hom_{\mathcal{C}}(Z, f)$ is continuous with respect to the topologies of $X^{\#}Z$ and $Y^{\#}Z$ for all Z in $\mathcal{C}$.

Definition 11 *We say that a topologized object* X *is* compact *if the corresponding functor* $X^{\#}$ *takes values in compact Hausdorff spaces. A category of compact objects and continuous morphisms in* $\mathcal{C}$ *will be denoted by* $C\mathcal{C}$

Lemma 12 *If* $g : S \to T$ *is a map in* CHo_* *then it is naturally a compact object in* Ho^2_**. In other words the categories* $(CHo_*)^2$ *and* CHo^2_* *have the same objects.*

PROOF. We need to show that for any $f : A \to B$ in Ho_* the set $\hom_{Ho^2_*}(f, g)$ has a natural compact topology. This is obvious since this set is the limit of the following diagram

$$[A,S] \times [B,T] \underset{\psi}{\overset{\varphi}{\rightrightarrows}} [A,T]$$

where the entries are compact since S and T are in CHo_*. The maps $\varphi(\alpha, \beta) = g\alpha$ and $\psi(\alpha, \beta) = \beta f$ are continuous. □

By adjointness argument we immediately obtain the following.

Lemma 13 *If T is in CHo_* then for any X the space $map_*(X,T)$ is in CHo_* and for any map $f : X \to Y$ the induced map $map_*(Y,T) \to map_*(X,T)$ is continuous.*

5 Localizations at a space

In this section we will prove (Theorem 19) that localization at a space Z exists whenever Z is a homotopy retract of a compact object in the sense of Definition 11. We attain this by showing that for such spaces Z any Z-equivalence can be presented as a filtered colimit of Z-equivalences of bounded cardinalities so that we can use Lemma 6.

Let $\mathcal{S}^2_*$ be the usual category of maps in $\mathcal{S}_*$. We will say that f_0 is a subobject of f if there is a cofibration $f_0 \hookrightarrow f$ and will denote this fact by $f_0 \subseteq f$. Given $f : A \to B$ we will write $|f|$ for the number of nondegenerate simplexes of $A \vee B$ and will say that f is finite if $|f|$ is.

Lemma 14 *Let $f \subseteq h$ be cofibrant objects in $\mathcal{S}^2_*$. Let g, fibrant in $\mathcal{S}^2_*$, represent an object in $Cho\mathcal{S}^2_*$. Let $\alpha \in \hom_{\mathcal{S}^2_*}(f,g)$. If for every finite subobject $k \subseteq h$ the map α extends to $f \cup k$ then α extends to h.*

PROOF. Let t be in $\mathcal{S}^2_*$ such that $f \subseteq t \subseteq h$. Let $r : \hom_{ho\mathcal{S}^2_*}(t,g) \to \hom_{ho\mathcal{S}^2_*}(f,g)$ be the restriction map. Define $E(t)$ as $r^{-1}([\alpha])$ that is the set of all extensions, in $ho\mathcal{S}^2_*$, of α to t. Since r is a continuous map between compact spaces we see that $E(t)$ is empty or compact. The limit $\lim E(f \cup k)$ taken over all finite subobjects of h is nonempty since it is directed and the sets $E(f \cup k)$ are compact (nonempty by assumption). The proof will be complete once we show that $E(h)$ is nonempty. We will show that $E(h) = \lim E(f \cup k)$. Let $\mathrm{map}_*(t,g)$ be a simplicial set whose n-simplexes form a set $\hom_{\mathcal{S}^2_*}(t \wedge (\Delta^n_+), g)$ and whose faces and degeneracies are induced by the cosimplicial structure on $\Delta^\bullet$. Obviously $\pi_0(\mathrm{map}_*(t,g)) = E(t)$. Since g represents an object in $Cho\mathcal{S}^2_*$ we see that $\pi_q(\mathrm{map}_*(t,g)) = \hom_{ho\mathcal{S}^2_*}(t \wedge (\Delta^q/\partial\Delta^1), g)$ is compact for $q \geq 0$ which gives us the last equation in the following sequence.

$$\pi_0(\mathrm{map}_*(h,g)) = \pi_0(\mathrm{map}_*(\mathrm{colim} f \cup k, g)) = \pi_0(\mathrm{map}_*(\mathrm{hocolim} f \cup k, g)) =$$

$$= \pi_0(\mathrm{holim}\ \mathrm{map}_*(f \cup k, g)) = \lim \pi_0(\mathrm{map}_*(f \cup k, g))$$

This means that

$$E(h) = \lim E(f \cup k)$$

□

Directly from Lemma 14 we obtain the following statement.

Lemma 15 *Given cofibrant f and fibrant g in $\mathcal{S}^2_*$ with g representing an object in $Cho\mathcal{S}^2_*$ there is a cardinal number $\tau = \tau(f,g)$ such that for any h in $\mathcal{S}^2_*$ with $f \subseteq h$ there is k in $\mathcal{S}^2_*$ such that $f \subseteq k \subseteq h$ and $|k| \leq \tau$ and if $\alpha : f \to g$ extends to $\alpha_k : k \to g$ then it extends to $\alpha_h : h \to g$.*

PROOF. For each $\alpha : f \to g$ which does not factor as $f \hookrightarrow h \to g$ Lemma 14 gives us a finite object k_α in $\mathcal{S}^2_*$ such that α does not factor as $f \hookrightarrow f \cup k_\alpha \to g$. We can take $k = f \cup \bigcup_\alpha k_\alpha$. Since each k_α is finite and the number of possible maps α depends only on f and g we see that there is an upper bound for the cardinality of k which depends only on f and g. $\square$

The role of this Lemma is following. We think of f and g as fixed and of h as uncontrollably big. We want the obstruction to extending a map from f to h to be detected on some k whose cardinality we can control.

Lemma 16 *Given cofibrant f and fibrant g in $\mathcal{S}^2_*$ with g representing an object in $Cho\mathcal{S}^2_*$ there is a cardinal number $\delta = \delta(f,g)$ such that for any h in $\mathcal{S}^2_*$ with $f \subseteq h$ there is k in $\mathcal{S}^2_*$ such that $f \subseteq k \subseteq h$ and $|k| \leq \delta$ and the restriction map $\hom_{ho\mathcal{S}^2_*}(h,g) \twoheadrightarrow \hom_{ho\mathcal{S}^2_*}(k,g)$ is an epimorphism.*

PROOF. The object k is constructed as a union of an ascending chain $f = k_0 \subseteq k_1 \subseteq ... \subseteq k_n \subseteq ....$ This chain is built by induction on n. Given k_n we use Lemma 15 to choose k_{n+1} so that $k_n \subseteq k_{n+1} \subseteq h$ and if a map $k_n \to g$ extends to k_{n+1} then it extends to h.

Given $\alpha : k \to g$ we need to show that we can extend α to $\tilde{\alpha} : h \to g$. By the construction of k there are maps $\alpha_n : h \to g$ such that $\alpha_n|_{k_n} \simeq \alpha|_{k_n}$. Since by assumption $\hom_{ho\mathcal{S}^2_*}(h,g)$ is compact we can take $\tilde{\alpha}$ to be an accumulation point of the set $\{\alpha_n\}$.

We have $\tilde{\alpha}|_{k_n} \simeq \alpha|_{k_n}$ for all n since the sequence $\alpha_i|_{k_n} \in \hom_{ho\mathcal{S}^2_*}(k_n,g)$ converges to $\alpha|_{k_n}$, it is actually constant for $i \geq n$, and the restriction map $\hom_{ho\mathcal{S}^2_*}(h,g) \to \hom_{ho\mathcal{S}^2_*}(k_n,g)$ is continuous.

A similar argument as in the last paragraph of the proof of Lemma 14 tells us that

$$\alpha \in \hom_{ho\mathcal{S}^2_*}(k,g) = \lim \hom_{ho\mathcal{S}^2_*}(k_n,g)$$

hence $\tilde{\alpha}|_{k_n} \simeq \alpha|_{k_n}$ for all n implies $\tilde{\alpha}|_k \simeq \alpha$. $\square$

Lemma 17 *Let g in $\mathcal{S}^2_*$ represent an object in $Cho\mathcal{S}^2_*$. Let cofibrant h and fibrant p be in $\mathcal{S}^2_*$. Let p be a retract in $\mathcal{S}^2_*$ of g and h have the homotopy LLP with respect to p. There is a cardinal $\gamma = \gamma(g)$ such that h is a colimit of* *subobjects h_α such that each h_α has the homotopy LLP with respect to p and $|h_\alpha| \le \gamma$.*

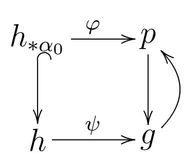
PROOF. We can write h as $h = \mathrm{colim} h_\alpha$ where each h_α is finite. Inductively we replace h_α with objects $h_{*\alpha}$ that have the left lifting property with respect to p. We start with the trivial object in $\mathbb{S}^2_*$, a map between spaces consisting of a basepoint only, which need not be replaced. Suppose that for some α_0 all subobjects of h_{α_0} have been replaced. Let $h' = h_{\alpha_0} \cup \bigcup_{\alpha<\alpha_0} h_{*\alpha}$. Lemma 16 gives us a factorization

$$h' \hookrightarrow h_{*\alpha_0} \hookrightarrow h$$

such that the restriction map

$$\hom_{ho\mathbb{S}^2_*}(h, g) \twoheadrightarrow \hom_{ho\mathbb{S}^2_*}(h_{*\alpha_0}, g) \tag{18}$$

is an epimorphism. We want to show that $h_{*\alpha_0}$ has the homotopy LLP with respect to p. For any map $\varphi : h_{*\alpha_0} \to p$ consider a diagram



where the map ψ exists by (18). Since by assumption h has the left lifting property with respect to p and any map from $h_{*\alpha_0}$ to p factors through h we obtain the homotopy LLP for $h_{*\alpha_0}$ with respect to p. We see that $|h_{*\alpha_0}|$ depends only on g, on $h_{*\alpha}$ for $\alpha < \alpha_0$ and on the bounds $\delta(h_{*\alpha}, g)$ from Lemma 16. □

We are ready to prove the main theorem of this paper. In the following we prefer to work in the Ho^2_* rather than in the equivalent category $ho\mathbb{S}^2_*$.

Theorem 19 *Let $\overline{Z}$ in Ho_* represent an object in CHo_*. For any Z in Ho_*, a homotopy retract of $\overline{Z}$, there exists a map f such that L_f is a localization at Z.*

PROOF. To use Lemma 8 we consider maps

$$p : \mathrm{map}_*((\bigvee_{n\ge 0} S^n)_+, Z) \to \mathrm{map}_*((\bigvee_{n\ge 0} S^n)_+, Z)$$

and

$$g : \mathrm{map}_*((\bigvee_{n\ge 0} S^n)_+, \overline{Z}) \to \mathrm{map}_*((\bigvee_{n\ge 0} S^n)_+, \overline{Z}).$$

We observe that p is a homotopy retract of g and by Lemma 13 g represents an object in CHo_*^2. By Lemma 8 a map h is a Z-equivalence if and only if it has the homotopy LLP with respect to p. By Lemma 17 there is a cardinal $\gamma = \gamma(g)$ such that any Z-equivalence h is a colimit of Z-equivalences whose cardinalities do not exceed γ. Since this is a directed colimit of cofibrations it is equivalent to a homotopy colimit. By Lemma 6 we can take f to be a wedge of all Z-equivalences whose cardinality does not exceed γ. □

Since one would like to remove the compactness assumption in Theorem 19 we briefly review the points where we used it in the proof. The key property we used in Lemmas 14 and 16 is that for a compactly topologized C and a directed diagram X_i in Ho_* there is a bijection

$$[\mathrm{holim} X_i, C] \overset{\simeq}{\to} \lim [X_i, C]$$

Other properties are much simpler, in Lemma 16 we needed to know that an infinite subset of a compact topological space has an accumulation point and in Lemma 12 that a closed subspace of a product of compact spaces is compact.

We end this section with Example 22 which shows that the "retract" condition in Theorem 19 is relevant. More precisely there are spaces which represent objects in CHo_* but whose retracts are not in CHo_*.

We will need the following two lemmas. By a *simplicial compact space* we understand a simplicial object in the category of compact (Hausdorff) topological spaces.

Lemma 20 *Let X be a simplicial set and Z a simplicial compact space. The set* $\hom_{\mathcal{S}_*}(X, Z)$ *has a natural compact topology.*

PROOF. To see this observe that $\hom_{\mathcal{S}_*}(X, Z)$ is a subset of

$$\prod_n \mathrm{Sets}(X_n, Z_n) \cong \prod_n \prod_{X_n} Z_n$$

which has a compact product topology. The subset $\hom_{\mathcal{S}_*}(X, Z)$ is determined by a number of equations (see May [15] 1.2) between continuous maps so it forms a closed hence compact subspace of the product. □

Lemma 21 *Let T be a simplicial compact space which is fibrant as a simplicial set. Then T naturally represents an object in CHo_*.*

PROOF. We need to show that for any simplicial set X the set $[X, T]$ is naturally compact. We have $\mathrm{map}_*(X, T)_k = \hom_{\mathcal{S}_*}(X \wedge (\Delta^k_+), T)$ hence by

Lemma 20 the mapping space $\mathrm{map}_*(X, T)$ is a simplicial compact space. Since $[X, T] = \pi_0 \mathrm{map}_*(X, T)$ hence by Proposition 4.7 in [5] it is naturally compact. $\square$

Example 22 *Let $n > 0$, $Z = K(\mathbb{Q}, n)$ and $\overline{Z} = K(S^1, n)$. As a model of $K(S^1, n)$ we use the one described in* 1.2 *of [3]; $K(S^1, n)_t$ is a product of $\binom{t}{n}$ copies of S^1, hence it is a compact topological space, faces and degeneracies are given by projections and group operations hence they are continuous. This model of $K(S^1, n)$ is a simplicial compact space which is fibrant as a simplicial set. It has a homotopy type of an Eilenberg-Mac Lane space for S^1 viewed as a discrete group. The group S^1 is a direct sum of $\mathbb{Q}/\mathbb{Z}$ and a rational vector space hence $\mathbb{Q}$ is a retract of S^1 and so Z is a retract of $\overline{Z}$. We have $\overline{Z}$ which represents an object in CHo_* and its retract Z which does not represent any objects in CHo_* since $\pi_n Z = \mathbb{Q}$ is an infinite countable group hence admits no compact structure.*

6 Applications and Examples

We note that Theorem 19 implies the existence of localizations at spaces which belong to the following classes:

a) Profinite completions of other spaces.
b) Simplicial compact spaces which are fibrant as simplicial sets (Lemma 21).
c) Mapping spaces with targets in a) or b) (Lemma 13).

Our first example of a localization at a space is an idempotent approximation to the profinite completion. The work of Rao [17] implies the existence of such an approximation defined on the nilpotent spaces. Here we don't require such assumptions.

The profinite completion was introduced by Sullivan in section 3 of [18] via the Brown representability theorem. To a given space X he assigns another space $\hat{X}$ which represents the functor $\hat{X}(Y) = \lim_{(X\downarrow\mathcal{F})}[Y, F]$. The limit is taken over the category $(X \downarrow \mathcal{F})$ whose objects are maps $X \to F$ in Ho_* with F connected and $\pi_q F$ finite for all $q > 0$. The morphisms are commutative diagrams in Ho_* as below.

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ F_1 & \longrightarrow & F_2 \end{array}$$

The functor $F : (X \downarrow \mathcal{F}) \to \mathcal{S}_*$ takes an object $X \to F_0$ to the space F_0. This limit is well defined since the category $(X \downarrow \mathcal{F})$ is equivalent to a small category.

Theorem 23 *There exists an idempotent approximation to the profinite completion. More precisely, there is the terminal localization among localizations L which admit the following factorization.*

$$X \to LX \to \hat{X}$$

PROOF. For each homotopy class of connected spaces with $\pi_q F$ finite for all $q > 0$ choose a representative F. Let $Z = \prod F$ be the product of those representatives. Since each F is naturally compact (in the sense of Definition 11) and $[Y, Z] = \prod[Y, F]$ for all Y we see that Z is compact. The localization L_Z exists by Theorem 19. We observe that if F is connected with $\pi_q F$ finite for $q > 0$ then F is Z-local. Let $r : Z \to F \hookrightarrow Z$ be the retraction onto the axis that corresponds to F. We see that $F \simeq \text{holim}(... \xrightarrow{r} Z \xrightarrow{r} Z)$ hence it is Z-local. This implies that $[L_Z X, F] \to [X, F]$ is a bijection and consequently that the categories $(X \downarrow \mathcal{F})$ and $(L_Z X \downarrow \mathcal{F})$ are equivalent hence $\hat{X} \simeq (L_Z X)\hat{}$ which leads us to the factorization we were looking for:

$$X \to L_Z X \to (L_Z X)\hat{} \simeq \hat{X} \tag{24}$$

It remains to show that L_Z is the terminal localization which admits factorization (24). Suppose that a localization T also admits (24). Since profinite completion is idempotent on finite spaces F as above we have

$$F \to TF \to \hat{F} \simeq F$$

so F is a homotopy retract of TF hence T-local. This means that the space Z is T-local hence by the definition of L_Z we have $T \leq L_Z$. □

Theorem 25 *Let h^* be a cohomology theory represented by an Ω-spectrum $\{\underline{h}_n\}$. If each $\underline{h}_n$ is a homotopy retract of a compact, in the sense of Definition 11, space then there exists a map f such that L_f-equivalences and h^*-equivalences coincide. In particular the corresponding cohomological localization exists.*

PROOF. Let $Z = \prod \underline{h}_n$ and use Theorem 19. □

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