LARGE LOCALIZATIONS OF FINITE GROUPS

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Abstract. We construct examples of localizations in the category of groups which take the Mathieu group M_{11} to groups of arbitrarily large cardinality which are "abelian up to finitely many generators". The paper is part of a broader study on the group theoretic properties which are or are not preserved by localizations.

MSC: **20J15**(20D99)

1. Introduction

Let $f : H \to G$ be a group homomorphism. We say (cf. [1]) that f is *closed* if it induces via composition a bijection of sets

$$\mathrm{Hom}(G, G) \to \mathrm{Hom}(H, G)$$

This paper is part of a broader study of the question regarding which group theoretic properties do and which do not pass from H to G via closed homomorphisms. It is more natural to formulate this in the language of localizations, as explained in the next paragraph.

The interest in closed homomorphisms is motivated by the following fact proved in [4, Lemma 2.1]. There exists a closed homomorphism $f : H \to G$ if and only if there exists a localization $L : Groups \to Groups$ such that $LH = G$. A *localization* (also called a reflection) is a functor $L : \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $\eta : Id \to L$ such that the compositions $L(\eta_X)$ and $\eta_{L(X)}$ are equal and are isomorphisms, for each object X. Thus the question we study is: which group theoretic properties are and which are not preserved by localizations. For a survey of this problem see [4].

In Section 7 we prove that for the Mathieu group M_{11} and any cardinal κ there exist closed inclusions $M_{11} \to G$ such that the cardinality of G is at least κ, G is generated by M_{11} and an abelian subgroup, and the abelianization of G is of the same cardinality as G, in particular G is very far from being simple. This example is obtained as a corollary of Theorem 6.4 which states conditions on a finite group S that imply the existence of such closed embeddings $S \to G$. We believe it should be possible to construct a solvable group S which meets these conditions.

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First examples of closed embeddings of a finite group into an infinite one were described by Libman [9]. Closed embeddings of finite simple groups into arbitrarily large simple groups were constructed by Göbel, Rodríguez and Shelah in [7] and [6]. Nonsimple localizations of finite simple groups were described by Rodríguez, Scherer and Viruel in [12]. Nonperfect localizations of infinite perfect groups were obtained independently by Badzioch and Feshbach in [2] and by Rodríguez, Scherer and Viruel in [13].

Closer to the "abelian end", Dwyer and Farjoun have asked whether a closed homomorphism $f : H \to G$ with H finite nilpotent must be an epimorphism. They proved that this is the case when H is of nilpotency class at most two (see [10, Theorem 3.3] and [4, Theorem 2.3]). Aschbacher [1] extended this result to the case where H is of class at most three, under the additional assumption that G is finite.

The author wishes to express his thanks to the referee for many suggestions, especially for removing a computer aided verification from the proof of Example 7.1 and two Properties from the list in Section 4

2. Bass-Serre theory

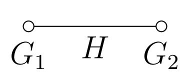
In this section we collect elements of the Bass-Serre theory on group amalgams acting on trees. Most of the results below can be found in [15], some in [11] and [3], others are simple corollaries. We follow the terminology of [15].

The neutral element of a group is always denoted by e. If k is an element and x is an element or a subset of a group we often write ${}^k x$ instead of kxk^{-1}.

If H is a common subgroup of G_1 and G_2 then the *amalgam* $G_1 *_H G_2$ is the colimit of the diagram $G_1 \supseteq H \subseteq G_2$, that is, the unique, up to isomorphism, group G which contains G_1 and G_2 and such that $G_1 \cap G_2 = H$ and for any pair of homomorphisms $f_1 : G_1 \to K$ and $f_2 : G_2 \to K$ if $f_1|_H = f_2|_H$ then there exists a unique homomorphism $f : G \to K$ extending both f_1 and f_2. It is customary to call G_1 and G_2 the *factors* of the amalgam $G_1 *_H G_2$.

Remark 2.1. The amalgam G is generated by $G_1 \cup G_2$.

Lemma 2.2. [15, §4.1 Theorem 7] *Let $G = G_1 *_H G_2$ be an amalgam of groups. There exists a unique, up to isomorphism, tree T_G on which G acts with fundamental domain a segment:*



The labels denote the stabilizers of the edge and its vertices. No element of G may swap the ends of an edge of T_G.

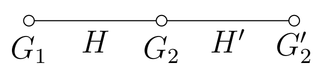
In this paper, as in [15], the group G acts on T_G from the left. Below we list some immediate consequences of 2.2.

Remark 2.3. (i) The stabilizers of the vertices of T_G are precisely the conjugates of either G_1 or G_2.

(ii) The vertex stabilizer acts transitively on the neighbors of that vertex (since G acts transitively on the edges of T_G).

(iii) The assignment of a stabilizer to a vertex of T_G is a one-to-one correspondence, which allows us to make no distinction between vertices and their stabilizers. This means that we identify the vertex set of T_G with $\{{}^gG_i \mid i = 1, 2, g \in G\}$ and the edge set of T_G with $\{\{{}^gG_1, {}^gG_2\} \mid g \in G\}$. Then G acts (on the left) on T_G via (left) conjugation. In particular, an x in G fixes a vertex V if and only if $x \in V$.

(iv) If the graph



is a fragment of T_G then, as in (ii), there exists an element $g \in G_2$ such that $G_1' = {}^gG_1$. Moreover for the same g we have $H' = {}^gH$ and $G_1 \cap G_1' = H \cap H'$.

The next lemma defines the notion of a *reduced decomposition* of an element $g \in G_1 *_H G_2$. We choose sets $R(G_1)$ and $R(G_2)$ of right coset representatives of $H\backslash G_1$ and $H\backslash G_2$ respectively such that e belongs to $R(G_1)$ and $R(G_2)$.

Lemma 2.4. *For every $g \in G_1 *_H G_2$ there exists a unique decomposition (called a reduced decomposition)*

$$g = ar_1r_2 \dots r_n$$

such that $a \in H$ and $r_i \in R(G_1) \cup R(G_2) \smallsetminus \{e\}$ for $i = 1, 2, \dots, n$ and for any $i = 1, 2, \dots, n-1$ one of r_i and r_{i+1} belongs to $R(G_1)$ and the other to $R(G_2)$.

Proof. See [15, §1.2, Theorem 1]. □

The integer n above is called the *length* of g and is denoted $l(g)$. It is easy to see that g decomposes as in Lemma 2.4 if and only if $g = g_0g_1 \dots g_n$, where $g_0 \in H$ and g_i, for $i = 1, 2, \dots n$, alternately belongs to $G_1 \smallsetminus H$ and $G_2 \smallsetminus H$. Therefore $l(g)$ does not depend on the choice of the right coset representatives and we have $l(g^{-1}) = l(g)$. An element $g = ar_1r_2 \dots r_n$ is called *cyclically reduced* if $l(g) \geq 2$ and one of r_1, r_n belongs to $G_1 \smallsetminus H$ and the other to $G_2 \smallsetminus H$. We note that $g \in G$ is cyclically reduced if and only if $l(g) \geq 2$ is even.

Lemma 2.5. *Every element of $G_1 *_H G_2$ is conjugate to a cyclically reduced one or to an element of $G_1 \cup G_2$. Every cyclically reduced element is of infinite order.*

Proof. See [15, §1.3 Proposition 2]. □

Lemma 2.6. *If g is a cyclically reduced element of $G_1 *_H G_2$ which is conjugate to an element of the form $r_1 r_2 \ldots r_k$, where $k \geq 2$ and r_i, r_{i+1} as well as r_1, r_k are in distinct factors then g can be obtained by cyclically permuting $r_1, r_2, \ldots r_k$ and then conjugating by an element of H.*

Proof. See [11, Theorem 4.6(iii)]. □

In particular, in view of the description of the length that follows Lemma 2.4, we have:

Lemma 2.7. *If g and h are conjugate cyclically reduced elements of $G_1 *_H G_2$ then $l(g) = l(h)$.*

If P, Q are vertices of a tree then $l(P, Q)$ is the length (i.e. the number of edges) of the shortest path from P to Q and is called the *distance* from P to Q. The shortest path is called a *geodesic*.

Lemma 2.8. *If $g = ar_1 r_2 \ldots r_n$ is a reduced decomposition of g in $G_1 *_H G_2$ with r_1 in G_1 then the geodesic from G_2 to ${}^{g^{-1}}G_2$ contains the vertices:*
G_2, ${}^{r_n^{-1}}G_1$, ${}^{r_n^{-1} r_{n-1}^{-1}}G_2$, $\ldots$, ${}^{r_n^{-1} r_{n-1}^{-1} \ldots r_1^{-1}}G_2$ if n is even, and
G_2, G_1, ${}^{r_n^{-1}}G_2$, ${}^{r_n^{-1} r_{n-1}^{-1}}G_1$, $\ldots$, ${}^{r_n^{-1} r_{n-1}^{-1} \ldots r_1^{-1}}G_2$ if n odd.

Proof. It is enough to observe that each r_i rotates the edge H about one of its end points and argue by induction: for $n = 0$ the claim is true. If n is odd then the case of $n-1$ implies that ${}^{r_n^{-1}}G_2$, ${}^{r_n^{-1} r_{n-1}^{-1}}G_1$, $\ldots$, ${}^{r_n^{-1} r_{n-1}^{-1} \ldots r_1^{-1}}G_2$ is a geodesic, and the claim follows since any path without backtracking in a tree is a geodesic. If n is even then, analogously, ${}^{r_n^{-1}}G_1$, ${}^{r_n^{-1} r_{n-1}^{-1}}G_2$, $\ldots$, ${}^{r_n^{-1} r_{n-1}^{-1} \ldots r_1^{-1}}G_2$ is a geodesic and the lemma follows. □

Lemma 2.9. *Suppose that $H_0 \subseteq H$ is such a subgroup that for x in $G_1 \cup G_2$ an inclusion ${}^x H_0 \subseteq H$ implies ${}^x H_0 = H_0$. Then the normalizer of H_0 in $G_1 *_H G_2$ may be presented as*

$$N_{G_1 *_H G_2}(H_0) = N_{G_1}(H_0) *_H N_{G_2}(H_0)$$

Proof. The right hand side is well defined since H_0 is normal in H. Only the inclusion $N_{G_1 *_H G_2}(H_0) \subseteq N_{G_1}(H_0) *_H N_{G_2}(H_0)$ is not obvious. Let $g = ar_1 r_2 \ldots r_n$ be the reduced decomposition of an element g in $N_{G_1 *_H G_2}(H_0)$. We have ${}^g H_0 = H_0 \subseteq G_1 \cap G_2$ hence

$$H_0 \subseteq {}^{g^{-1}}G_1 \cap {}^{g^{-1}}G_2 \cap G_1 \cap G_2$$

and therefore the elements of H_0 fix all the vertices of the geodesics which connect G_2 or G_1 to ${}^{g^{-1}}G_2$ or ${}^{g^{-1}}G_1$. By Lemma 2.8 we have $H_0 \subseteq {}^{r_n^{-1} r_{n-1}^{-1} \dots r_i^{-1}}G_2$ and $H_0 \subseteq {}^{r_n^{-1} r_{n-1}^{-1} \dots r_i^{-1}}G_1$ for $i = 1, 2, \dots, n$, hence ${}^{r_i r_{i+1} \dots r_n}H_0 \subseteq G_1 \cap G_2 = H$ and therefore, by a downward induction, $r_i \in N_{G_1}(H_0) \cup N_{G_2}(H_0)$ for $i = n, n-1, \dots, 1$. □

Lemma 2.10. *Suppose that $G = E *_H K$ is such that if $x, y \in H$ are conjugate in E then they are conjugate in K. Then if $x, y \in K$ are conjugate in G then they are conjugate in K.*

Proof. See [11, Theorem 4.6 (i) and (ii)]. □

Lemma 2.11. *If $g \in G = G_1 *_H G_2$ then the action of g on T_G satisfies one of the following:*

a) *g has no fixed points.*
b) *g fixes a unique vertex of T_G.*
c) *g fixes a conjugate of G_1 and a conjugate of G_2.*

Proof. If a) and b) do not hold then g fixes two points P and Q. By the uniqueness it has to fix the geodesic from P to Q. The proof is complete since the vertices of every path are conjugates of G_1 and G_2 alternately. □

Lemma 2.12. *Let $F \subseteq G_1 *_H G_2$ be a finite subgroup. Then F is conjugate to a subgroup of G_1 or G_2.*

Proof. See [3, Chapter II, Corollary A3]. □

A doubly infinite chain

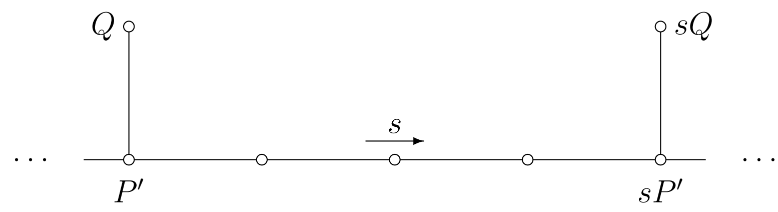
is called a *straight path*.

Lemma 2.13. *Let s be an automorphism acting on a tree X with no fixed points. Let*

$$m = \inf_{P \in \operatorname{vert} X} l(P, sP) \quad \text{and} \quad T = \{P \in \operatorname{vert} X \mid l(P, sP) = m\}$$

Then:

i) *T is the vertex set of a straight path of X.*
ii) *s induces a translation of T of amplitude m.*
iii) *Every subtree of X stable under s and s^{-1} contains T.*
iv) *If a vertex Q of X is at a distance d from T then $l(Q, sQ) = m + 2d$.*



$(m = 4, d = 1)$

Proof. See [15, §6.4, Proposition 24]. □

Lemma 2.14. *Let g be a cyclically reduced element of $G = G_1 *_H G_2$. Let $X = T_G$ and m and T be as in Lemma 2.13, where s is the action of g on T_G. Then $l(g) = m$ and both G_1 and G_2 belong to T.*

Proof. Let $g = ar_1r_2 \dots r_n$ be the reduced decomposition of g. Possibly swapping G_1 and G_2, we may assume that $r_1 \in G_1$. Lemma 2.8 implies that $l(g) = l(G_2, {}^{g^{-1}}G_2)$. Let $[G_2, {}^{g^{-1}}G_2]$ be the geodesic from G_2 to ${}^{g^{-1}}G_2$. Let $T_0 = \bigcup_{k\in\mathbb{Z}} {}^{g^k}[G_2, {}^{g^{-1}}G_2]$. The union of consecutive geodesics $[{}^{g^{k+2}}G_2, {}^{g^{k+1}}G_2] \cup [{}^{g^{k+1}}G_2, {}^{g^k}G_2]$ is a path of length $2l(g)$ that connects ${}^{g^k}G_2$ and ${}^{g^k}({}^{g^2}G_2)$. Since g is cyclically reduced we see that $l(g^2) = 2l(g)$. Lemma 2.8 implies that the geodesic connecting ${}^{g^k}G_2$ and ${}^{g^k}({}^{g^2}G_2)$ has length $l(g^2)$ hence the path above is a geodesic and consequently, since T_G is a tree, the union T_0 is a straight path on which g acts as translation by $l(g)$. Since T_0 is invariant under g and g^{-1}, Lemma 2.13(iii) implies that $T \subseteq T_0$, so since they are both straight paths, we have $T = T_0$, hence $G_2 \in T$ and $l(g) = m$. We prove that $G_1 \in T$ by repeating the argument above with g replaced by g^{-1} and G_1 swapped for G_2. □

3. Large E-rings

This section describes the "filling", which makes targets of our closed embeddings arbitrarily large. The reader interested in "infinite" rather than "arbitrarily large" may take $E = \mathbb{Z}_{(q)} = \{\frac{m}{n} \in \mathbb{Q} \mid q \nmid n\}$.

The notion of an E-ring was introduced by Schultz [14]. Let $\mathrm{Hom}_R(E, E)$ denote the ring of endomorphisms of E as a right R-module. A ring E with identity 1 is said to be an *E-ring* if the ring restriction homomorphism

$$E \cong \mathrm{Hom}_E(E, E) \to \mathrm{Hom}_{\mathbb{Z}}(E, E)$$

is an isomorphism. This forces E to be commutative. Unless explicitly stated, we work only with the additive group of E and denote it with the same symbol E.

With the terminology outlined in Section 1 above we may characterize the additive groups of E-rings as those which admit a nontrivial closed homomorphism $\mathbb{Z} \to E$, or equivalently, as possible values of group localizations of the integers.

We make use of a particular class of examples of E-rings, constructed by Dugas, Mader and Vinsonhaler [5]. The following theorem is extracted from [5].

Theorem 3.1. *For any prime number q and an infinite cardinal number κ, not strictly between $\aleph_0$ and the continuum, there exists an abelian group E of cardinality κ with the following properties:*

(1) *E is the additive group of an E-ring.*
(2) *E is torsion free.*
(3) *No nonzero element of E is divisible by all powers of q.*
(4) *E is p-divisible for every prime $p \neq q$.*
(5) *All nontrivial endomorphisms $f : E \to E$ are injective.*

Item (5) of Theorem 3.1 is not explicitly stated in [5] but E is constructed as a ring with no zero divisors (i.e. elements $a \neq 0$, $b \neq 0$ such that $ab = 0$). Since each endomorphism of an E-ring is a multiplication by an element of E, item (5) follows.

4. Construction of closed embeddings

In this section we construct arbitrarily large groups L and closed embeddings $S \to L$ with finite source S.

Let S be a finite group with no outer automorphisms. Let a and b be two elements of S. Let $A = \langle a \rangle$ be the cyclic subgroup of S generated by a and let $N = N_S(A)$ be its normalizer in S. We fix a prime number $p > 2$ and assume the following.

Properties:

P1. The order of a is p.
P2. The element b is not in N.
P3. $b^2 = e$.
P4. If $k \in S$ commutes with a and b then $k = e$.
P5. Any homomorphism $f : S \to S$ is either an automorphism or contains a and b in its kernel.
P6. S contains no element of order p^2.
P7. p does not divide the order of N/A.
P8. The intersection $N \cap {}^bN$ is trivial.

Remark 4.1. Property P8 implies that for any $k \in N$ if $kbk^{-1}b^{-1} \in A$ then $k = e$.

Remark 4.2. Property P7 and the Schur-Zassenhauss Theorem imply that the exact sequence $\{e\} \to A \to N \to N/A \to \{e\}$ splits.

Remark 4.3. For any infinite cardinal number κ, not strictly between $\aleph_0$ and the continuum, there exists an E-ring E, as in Theorem 3.1, which is p-divisible for every prime p dividing the order of S. To obtain such an E it is enough to choose the prime q so that it does not divide the order of S and use Theorem 3.1(4).

Let $C = \langle c \rangle$ be a cyclic group of order p^2. We identify A with the subgroup of C generated by c^p. The restriction homomorphism $\mathrm{Aut}(C) \to \mathrm{Aut}(A)$ splits uniquely, hence the split exact sequence in Remark 4.2 extends to a split sequence:

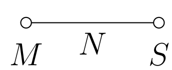
$$\{e\} \to C \to M \to M/C \to \{e\}$$

where $M/C \cong N/A$. We view N as a subgroup of M. We note that since $p > 2$ the unique split of $\mathrm{Aut}(C) \to \mathrm{Aut}(A)$ takes $a \mapsto a^{-1}$ to $c \mapsto c^{-1}$.

Let

$$K = M *_N S$$

be an amalgam of groups. By 2.2 there exists a unique, up to isomorphism, tree T_K on which K acts with fundamental domain



where the labels denote the stabilizers of the edge and its vertices.

In a similar way we define

$$L = E *_Z K$$

where the group E is chosen as in Remark 4.3 and $Z \subseteq E$ is the subgroup generated by the ring identity $1 \in E$ and is identified with $\langle cb \rangle \subseteq K$. Let T_L be the tree which corresponds to the amalgam L. We denote by $\eta : S \to L$ the inclusion of S into L. The remainder of this paper is devoted to the proof that η is closed and that M_{11} satisfies Properties P1–P8.

5. Properties of the construction

In this section we describe some properties of the inclusion $\eta : S \to L$, introduced in Section 4, which are used in Section 6.

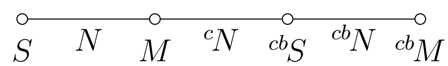
Remark 5.1. By the construction, M is generated by $N \cup \{c\}$. Remark 2.1 implies that the group K is generated by $S \cup \{c\}$. The group L is generated by K and E and therefore by S and E.

Lemma 5.2. *Let $g = (cb)^v$ for some $v > 0$. The normalizer of $\langle g \rangle$ in K is $\langle cb \rangle$.*

Proof. Since $c \in M \smallsetminus N$ and $b \in S \smallsetminus N$ we see that g is cyclically reduced of length $2v$ in $K = M *_N S$. Lemmas 2.13 and 2.14 imply that there exists a unique straight path $T \subseteq T_K$, stable under the action of $\langle g \rangle$; moreover M and S belong to T and the action of g restricted to T is a translation of amplitude $l(g) = 2v$.

If $k \in N_K(\langle g \rangle)$ then $kgk^{-1} = g^{\varepsilon}$ for some $\varepsilon \in \{-1, 1\}$, hence kgk^{-1} stabilizes the unique path T above and therefore ${}^{gk^{-1}}T = {}^{k^{-1}}T$, hence again by the uniqueness of T as in Lemma 2.13(iii) we have ${}^{k^{-1}}T = T$. We conclude that $N_K(\langle g \rangle)$ stabilizes T.

Let $\phi : N_K(\langle g \rangle) \to \mathrm{Aut}(T)$ be the homomorphism obtained by restricting the automorphisms of T_K to automorphisms of $T \subseteq T_K$. In this paragraph we prove that ϕ is one-to-one. We draw a part of T:



Since $\ker \phi$ acts trivially on T we have $\ker \phi \subseteq {}^cN \cap {}^{cb}N$. Property P8 implies that $N \cap {}^bN = \{e\}$, hence $\ker \phi \subseteq {}^c(N \cap {}^bN) = \{e\}$.

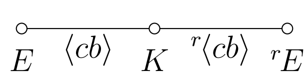
Since $\langle cb \rangle \subseteq N_K(\langle g \rangle)$, the action of $N_K(\langle g \rangle)$ on the vertices of T has two orbits: the S conjugates and the M conjugates, and therefore it is enough to prove that no element of $N_K(\langle g \rangle)$ acts on T as a reflection at S. Suppose to the contrary that $x \in N_K(\langle g \rangle)$ is such an element. Then $x \in S$ since x fixes S, and $x^2 = e$ since $\ker \phi$ is trivial. Also $xcbx^{-1} = (cb)^{-1}$, hence $cbxcbx = e$. Since cbx is torsion Lemma 2.12 implies that it belongs to a conjugate of S or M. Since $bx \in S$ we have $l(cbx) \leq 2$, hence $cbx \in S \cup M$. Since $c \in M \smallsetminus N$ we have $cbx \in M$, hence $bx \in S \cap M = N$. Remark 4.2 implies that $N = A \rtimes (N/A)$, hence $bx = a_0 n$ for some $a_0 \in A$ and $n \in N/A$. Since $e = (cbx)^2 = (ca_0 n)^2$ and $M = C \rtimes (N/A)$ we have $n^2 = e$; hence, as $N = A \rtimes (N/A)$, also $(bx)^2 = (a_0 n)^2 \in A$.

If $(bx)^2 = e$ then, since $x^2 = e$ and $b^2 = e$ (by Property P3), we see that b commutes with $bx \in N$. Since $c^2 \neq e$ and $(cbx)^2 = e$ we see that $bx \neq e$. This contradicts Property P8.

If $(bx)^2 \neq e$ then since the order of A is a prime p we see that $(bx)^2$ generates A. However b inverts $(bx)^2$ and hence normalizes A, contradicting Property P2. ☐

Lemma 5.3. *For $k \in L$, if $E \neq {}^kE$ then $E \cap {}^kE = \{e\}$.*

Proof. The path that connects E and ${}^kE$ must contain a segment conjugate to



for some $r \in K \setminus \langle cb \rangle$. Up to conjugation, we have $E \cap {}^kE \subseteq \langle cb \rangle \cap {}^r\langle cb \rangle$. If $\langle cb \rangle \cap {}^r\langle cb \rangle$ is nontrivial then we have $(cb)^s = r(cb)^t r^{-1}$ for some nonzero integers s and t. Lemma 2.7 applied to $K = M *_N S$ implies that $|s| = |t|$, hence $r \in N_K(\langle (cb)^s \rangle)$. Lemma 5.2 implies that $N_K(\langle (cb)^s \rangle) = \langle cb \rangle$, which contradicts $r \notin \langle cb \rangle$. □

Lemma 5.4. *If $g \in L$ normalizes A then $g \in K$.*

Proof. Since $a \in K$ its action on T_L fixes K. Since a is torsion and E is torsion-free Lemma 2.11 implies that K is the unique fixed point of this action. Since g normalizes A we see that ${}^{gag^{-1}}K = K$, hence ${}^{ag^{-1}}K = {}^{g^{-1}}K$ and therefore ${}^{g^{-1}}K = K$ by the uniqueness of the fixed point of a. The identity ${}^{g^{-1}}K = K$ implies $g \in K$. □

Lemma 5.5. *The image of a homomorphism $f : E \to L$ is conjugate in L to a subgroup of E.*

Proof. We need to prove that the action of E on T_L induced by f has a fixed vertex that corresponds to a conjugate of E. Suppose that $f(E)$ is nontrivial. Let $x \in E \setminus \ker f$. If x acts on T_L without fixed points then by Lemma 2.13 there exists a straight path T in T_L, stable under the action of x, on which it induces a translation by $n > 0$ vertices. Theorem 3.1(4) implies that E is divisible by many primes, hence there exist $y \in E$ and an integer $m > n$ such that $y^m = x$. The action of y on T_L also has no fixed points and by Lemma 2.13 again, there exists a straight path T_y stable under the action of y, which y translates by n_y vertices. Since T_y is stable under x and x^{-1}, Lemma 2.13(iii) implies that $T \subseteq T_y$, so since they are both straight paths, $T = T_y$. Now n_y has to be a fractional quantity $\frac{n}{m}$, so we obtain a contradiction and therefore the set T^x of points fixed by x is nonempty.

Lemma 5.3 implies that T^x may contain at most one vertex of the form ${}^kE$ for some k. If T^x does contain a ${}^kE$ then since E is abelian we deduce that T^x is stable under the action of E on T_L, hence ${}^kE$ is fixed by this action, that is, $f(E) \subseteq {}^kE$ as required. If T^x does not contain a conjugate of E then it consists of precisely one conjugate of K and, as above, we have $f(E) \subseteq {}^kK$.

Let m be the order of S. Since E was chosen in Remark 4.3 to be m-divisible and no nontrivial element of K is divisible by all powers of m, we obtain a contradiction with the assumption that $x \notin \ker f$. □

Lemma 5.6. *If $f : L \to L$ is a homomorphism such that $f(b) = e$ then $f(L) = f(S)$ and f is uniquely determined by its values on S.*

Proof. We see that if 1 is the ring identity of E then $f(1) = f(cb) = f(c)$, hence $f(1)$ is torsion. Since E is torsion free, Lemma 5.5 implies that $f(1) = e$ and therefore $f(E) = \{e\}$ since any endomorphism of

E is either trivial or injective (Theorem 3.1(5)). Our claim is proved since L is generated by S and E. $\square$

Lemma 5.7. *For any homomorphism $f : S \to L$ there exists an inner automorphism c_g of L such that $c_g f(S) \subseteq S$, that is, for some $f_S : S \to S$, the following diagram commutes:*

$$\begin{array}{ccc} S & \xrightarrow{f} & L \\ \downarrow{\scriptstyle f_S} & & \downarrow{\scriptstyle c_g} \\ S & \xrightarrow{\eta} & L \end{array}$$

where $\eta : S \to L$ is the inclusion.

Proof. Since S is finite Lemma 2.12 tells us that its action on T_L has a fixed point. Hence, as there are no nontrivial homomorphisms $S \to E$, we see that $f(S)$ is conjugate to ${}^k f(S) \subseteq K$. The group K again acts on a tree T_K, hence by Lemma 2.12 again we see that ${}^k f(S)$ is conjugate to a subgroup of S or M.

It is enough to show that if $f(S)$ is isomorphic to $G \subseteq M$ then G is conjugate in M to a subgroup of N. Let H be the centralizer of A in N. Then

$$M/H \cong (C/A) \rtimes (N/H)$$

Now N/H is isomorphic to a subgroup of $\mathbb{Z}/(p-1)$. Notice that if $x \in M$ is an element of order p, then $x \in H$ (actually, by Property P7, $x \in A$, but we do not need this). Since by Property P6, G contains no elements of order p^2, the image of G in M/H is a p'-group. Hence, by Hall's theorem, this image is conjugate in M/H to a subgroup of N/H, and it follows that G is conjugate in M to a subgroup of N. $\square$

6. Proof of the main theorem

Proposition 6.1. *For any homomorphism $f : S \to L$ there exists a homomorphism $f' : L \to L$ which closes the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\eta} & L \\ & {\scriptstyle f}\searrow & \swarrow{\scriptstyle f'} \\ & L & \end{array}$$

where η is the inclusion.

Proof. It is enough to prove our claim for f composed with some automorphism of L; hence by Lemma 5.7 we may assume that $f = \eta f_S$, so it is enough to close the following diagram:

$$\begin{array}{ccc} S & \xrightarrow{\eta} & L \\ \Big\downarrow{\scriptstyle f_S} & & \Big\downarrow{\scriptstyle f'} \\ S & \xrightarrow{\eta} & L \end{array}$$

If f_S is an automorphism then it is an inner automorphism, hence we define f' as the inner automorphism of L determined by the same element. Otherwise by Property P5 we have $f_S(a) = e = f_S(b)$, hence $f_S(A) = \{e\}$. Since, by definition, $N/A = M/C$ we see that f_S restricted to N extends to $f' : M \to L$ so that $f'(c) = e$ and we obtain an extension $f' : K = M *_N S \to L$. Since $f'(cb) = e$ we extend it further to E by defining $f'(E) = \{e\}$. □

Lemma 6.2. *If a homomorphism $f : L \to L$ is the identity on S then f is the identity on L.*

Proof. Since f is the identity on $A \subseteq S$ the kernel of f must trivially intersect C so that the order of $f(c)$ is p^2. Since no element of E or S has order p^2, Lemma 2.12 implies that $f(c)$ is conjugate in L to a generator of C. We have

$$f(c) = gc^m g^{-1}$$

for some $g \in L$ and m not divisible by p. Since $a = c^p$ we see that $a = f(a) = ga^m g^{-1}$, that is, g normalizes $\langle a \rangle$, hence by Lemma 5.4 $g \in K = M *_N S$.

Since A is the unique subgroup of N of order p, Lemma 2.9 implies that $g \in N_M(A) *_N N_S(A) = M *_N N = M$, hence g normalizes C. Possibly changing the value of m, we obtain

$$f(c) = c^m$$

Since $cb \in E$, Lemma 5.5 implies that $f(cb) \in {}^kE$ for some $k \in L$. Let $\pi : E *_Z K \to E/Z$ be the amalgamation of the projection $E \to E/Z$ and a homomorphism which sends K to e. Since c and b are in K we see that

$$f(cb) \in {}^kE \cap \ker \pi$$

but the kernel of π restricted to ${}^kE$ is ${}^k\langle cb \rangle$, hence

$$c^m b = f(cb) = k(cb)^n k^{-1}$$

for some $n \in \mathbb{Z}$ and $k \in L$. Since $c^m b$ and $(cb)^n$ belong to K, Lemma 2.10 implies that we may choose k to be in K.

Lemma 2.6, applied to $c^m b = k(cb)^n k^{-1}$, implies that $n \in \{-1, 1\}$, and since $b^{-1} = b$, we have

$$c^m b = kc^n bk^{-1}$$

where $k \in N$. Therefore

$$c^m = (kc^nk^{-1})(kbk^{-1}b^{-1})$$

Since c^m and kc^nk^{-1} belong to C we have $kbk^{-1}b^{-1} \in C \cap S = A$, and therefore Remark 4.1 implies $k = e$, so $m = n$, that is, $f(c) = c$ or $f(c) = c^{-1}$. Since the latter implies $a = f(a) = a^{-1}$, which is impossible by Property P1, we have

$$f(c) = c$$

Since K is generated by S and c we see that f is an identity on K.

By Lemma 5.5 we know that f maps E to ${}^kE$ for some $k \in L$. Since $cb \in K$ we know that $f(cb) = cb$, hence $cb \in E \cap {}^kE$. Thus $E = {}^kE$ by Lemma 5.3. The kernel of $f - \mathrm{id}$ restricted to E contains cb, hence by Theorem 3.1(5) we see that f is the identity on E. □

Proposition 6.3. *If $f, g : L \to L$ are two homomorphisms that coincide on S then they are equal.*

Proof. If the kernel of f intersects S nontrivially then by Lemma 5.7 and Property P5 we have $g(b) = f(b) = e$, hence $f = g$ by Lemma 5.6.

If the kernel of f intersects S trivially then by Lemma 5.7 there exists an automorphism $h : L \to L$ such that hf induces an automorphism of S. Since S has no outer automorphisms, we may assume that hf is the identity on S. Since hf and hg are identities on S our claim follows from Lemma 6.2. □

The following theorem summarizes the results of the paper.

Theorem 6.4. *If a finite group S has no outer automorphisms and satisfies Properties P1–P8 of Section 4 then for any cardinal number κ there exists a closed inclusion $S \to L$ such that:*

1. *The cardinality of L is not less than κ.*
2. *There exists an abelian subgroup $E \subseteq L$ such that L is generated as a group by E and the image of S in L.*
3. *There exists an epimorphism $\pi : L \to E/Z$ where Z is an infinite cyclic subgroup of E. The composition $S \to L \to E/Z$ is trivial.*

Proof. Propositions 6.1 and 6.3 imply that the inclusion $\eta : S \to L$ described in Section 4 is closed. Items (1)–(3) follow immediately from the construction, presented in Section 4. □

7. Example: closed embeddings of the Mathieu group.

In this section we prove, as a consequence of Theorem 6.4, the existence of closed embeddings of the Mathieu group M_{11} into arbitrarily large groups.

Example 7.1. *Let $S = M_{11}$ be the Mathieu group. For any cardinal number κ there exists a closed embedding $M_{11} \to L$ such that the cardinality of L is at least κ. The group L is generated by M_{11} and an abelian subgroup $E \subseteq L$. The group L has an abelian quotient isomorphic to E/Z where Z is an infinite cyclic subgroup of E, in particular L is far from being simple.*

Proof. We use [8, p. 262]. Let a be an element of order 11 in $S = M_{11}$. Then $N := N_S(\langle a \rangle)$ is a group of order $11 \cdot 5$ and $C_S(a) = \langle a \rangle$. Let $b \in S$ be an involution. Then Properties P1–P7 are immediate. Further if we choose b so that b does not normalize any 5-Sylow subgroup of N (the existence of such a b follows from an easy counting argument), then property P8 holds (since b normalizes $N \cap {}^bN$). □

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