EVERY COUNTABLE GROUP IS THE FUNDAMENTAL GROUP OF SOME COMPACT SUBSPACE OF $\mathbb{R}^4$

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Abstract. For every countable group G we construct a compact path connected subspace K of $\mathbb{R}^4$ such that $\pi_1(K) \cong G$. Our construction is much simpler than the one found recently by Virk.

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The purpose of this note is to present a shorter proof of the following.

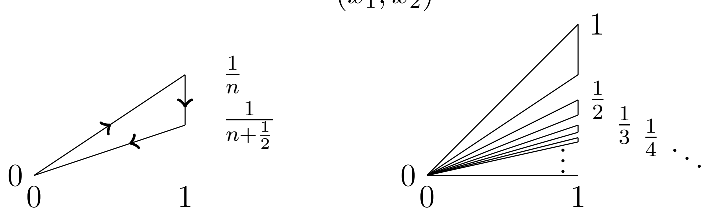
Theorem 1 (Virk [2, Theorem 2.20])**.** *For every countable group G there exists a compact path connected subspace $K \subseteq \mathbb{R}^4$ such that $\pi_1(K) \cong G$.*

This answered a problem posed by Pawlikowski [1]. The context and motivation was nicely outlined by Virk so we confine ourselves to presenting the proof.

Let $I = [0, 1]$ be the unit interval. We construct K inside I^4. We describe it by drawing two dimensional sections and projections of components of K. If (x_1, x_2, x_3, x_4) parametrize I^4 then (x_i, x_j) above every set of figures below indicate which coordinates parametrize the sections shown, the first being always horizontal. The remaining two coordinates are constant.

Let $(0, 0, 0, 0)$ be the basepoint. Define a loop g_n as the boundary of the triangle, as in Figure (A), with vertices $(0, 0, 0, 0)$, $(1, \frac{1}{n}, 0, 0)$ and $(1, \frac{1}{n+\frac{1}{2}}, 0, 0)$.

(x_1, x_2)



(A) loop g_n in $I^2 \times \{0\}^2$ (B) V in $I^2 \times \{0\}^2$

Let

$$V = I \times \{0\}^3 \cup \bigcup_{n \geq 1} g_n$$

be the union of the loops g_n for $n \geq 1$ and the interval $I \times \{0\}^3$. Then V is closed and its fundamental group $\pi_1(V)$ is a free group. The elements of its basis are represented by the loops g_n for $n \geq 1$. We use the same symbols g_n to denote the corresponding elements of the fundamental group.

Lemma 2. *Every countable group G can be presented as*

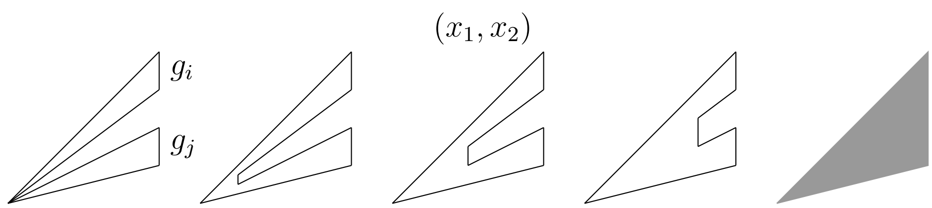
$$G = \langle g_1, g_2 \ldots \mid r_1, r_2, \ldots \rangle$$

where

(a) *Each relation r_n is of the form $g_i g_j g_k$, $g_i g_j$ or g_i with $i < j < k$.*
(b) *Each generator g_n appears in finitely many relations r_i.*

Proof. Let $(g_n)_{n=1}^{\infty}$ be a sequence in which every element of G occurs infinitely many times. We add relations as follows. Whenever $g_i = e$ in G we add a relation g_i. For every i we choose the least j such that $i < j$ and $g_i = g_j^{-1}$, and we add $g_i g_j$. Finally, for every identity $ab = c$ in G we choose g_i, g_j and g_k which, so far, have not appeared in relations of the third type, $i < j < k$, $g_i = a$, $g_j = b$ and $g_k = c^{-1}$. We add $g_i g_j g_k$. These relations reconstruct the multiplication table of G and each generator appears in at most four relations. We label the relations by natural numbers in any way we want. □

Fix a presentation of G as in Lemma 2. For every relation r_n we define a subset $R_n \subseteq I^2 \times \{\frac{1}{n}\} \times I$. We illustrate the case $r_n = g_i g_j$ by drawing several sections of R_n with planes of the form $I^2 \times \{\frac{1}{n}\} \times \{\frac{k}{4}\}$ where $k = 0, 1, 2, 3, 4$. The other cases are analogous. Note that the proportions are distorted in order to improve readability of the drawing.



$R_n \cap I^2 \times \{\frac{1}{n}\} \times \{0\}$ $R_n \cap I^2 \times \{\frac{1}{n}\} \times \{\frac{1}{4}\}$ $R_n \cap I^2 \times \{\frac{1}{n}\} \times \{\frac{1}{2}\}$ $R_n \cap I^2 \times \{\frac{1}{n}\} \times \{\frac{3}{4}\}$ $R_n \cap I^2 \times \{\frac{1}{n}\} \times \{1\}$

We first give a not yet correct anticipation of the construction: Let $W' = V \times I \times \{0\}$ and let $K' = W' \cup \bigcup_{n \geq 1} R_n$. Then $\pi_1(W') \cong \pi_1(V)$ is the free group with basis $\{g_n\}_{n \geq 1}$. Since W' is a neighborhood deformation retract of K' we may use Seifert–van Kampen's Theorem to see that $\pi_1(K') \cong G$. However K' is not closed, and its closure wrecks its fundamental group. We need a bit skinnier replacement for W':

Define $m : \mathbb{N} \longrightarrow \mathbb{N}$ by letting $m(n)$ be the least integer such that g_n appears in no r_m for $m > m(n)$. Then m is well defined by Lemma 2(b).

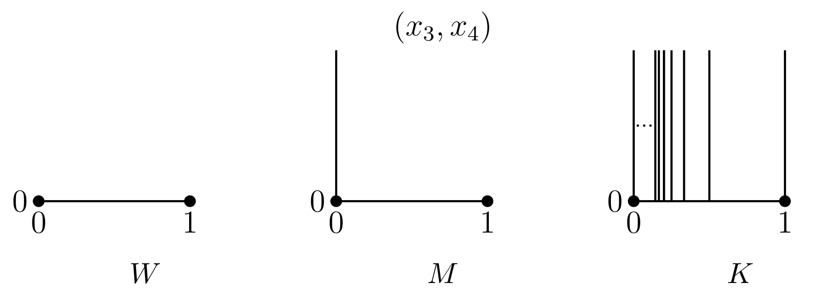
If we look at each loop $g_n \subseteq V$ separately we see that we do not need to "fatten" it to the whole of $g_n \times I \times \{0\}$. In order to attach all the relevant relations, it is enough to take $g_n \times [\frac{1}{m(n)}, 1] \times \{0\}$. Thus we define

$$W'' = I \times \{0\} \times I \times \{0\} \cup \bigcup_{n \geq 1} g_n \times \left[\frac{1}{m(n)}, 1\right] \times \{0\}$$

We see that W'' is closed. Still $\pi_1(W'') \cong \pi_1(V)$. Our gain is that the intersection $W'' \cap I^2 \times \{0\} \times I = I \times \{0\}^3$ is now contractible so that if $W = W'' \cup I^2 \times \{0\}^2$ then $\pi_1(W) \cong \pi_2(W'') \cong \pi_1(V)$ is still the free group on $\{g_n\}_{n \geq 1}$. Define

$$M = I^2 \times \{0\} \times I \cup W \quad \text{and} \quad K = M \cup \bigcup_{n \geq 1} R_n \,.$$

Below we show the projections of W'', M and K onto $\{0\}^2 \times I^2$:



The left solid dots represent $I^2 \times \{0\}^2$. The right ones represent V. The left vertical wall in the central figure is the "compactification wall", $I^2 \times \{0\} \times I$. Clearly W is a deformation retract of M, hence $\pi_1(M) \cong \pi_1(V)$ is free on g_n's. In the right figure we add the R_n's. The closedness of W implies the closedness of K. Since W is a neighborhood deformation retract of K we use Seifert–van Kampen's Theorem to prove that $\pi_1(K) \cong G$. Theorem 1 is proved.

We leave it to the reader to notice that the construction above generalizes to the following.

Exercise 3. For every positive integer n and a countable Abelian group G there exists a compact path connected subspace $K \subseteq \mathbb{R}^{n+3}$ such that $\pi_n(K) \cong G$.

References

[1] J. Pawlikowski, *The fundamental group of a compact metric space*, Proc. Amer. Math. Soc. 126 (1998), 3083–3087.

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