MEASURABLE CARDINALS AND FUNDAMENTAL GROUPS OF COMPACT SPACES

ADAM J. PRZEŹDZIECKI[1]

Institute of Mathematics, Warsaw University, Warsaw, Poland
Email address: adamp@mimuw.edu.pl
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Abstract. We prove that that all groups can be realized as fundamental groups of compact spaces if and only if no measurable cardinals exist. If the cardinality of a group G is nonmeasurable then the compact space K such that $G = \pi_1 K$ may be chosen so that it is path connected.

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We construct a group whose cardinality equals the least measurable cardinal and which cannot be realized as the fundamental group of a compact Hausdorff space (Theorem 4.2). Since Keesling and Rudyak proved [6] that every group of smaller cardinality is the fundamental group of some compact space, we see that the large cardinal axiom about existence of measurable cardinals is equivalent to the statement that all groups can be obtained as fundamental groups of compact spaces. The last section gives an affirmative answer to the question, asked in [6], whether each group of nonmeasurable cardinality is the fundamental group of a path connected compact space.

All spaces considered below are completely regular, I is the closed interval $[0, 1]$ and S^1 is a circle. If X is a space then we denote its Stone-Čech compactification by βX and its Hewitt realcompactification by υX. If $f : X \to Y$ is a map then $\overline{f}$ denotes the induced map between the compactifications: $\beta X \to \beta Y$ or $\upsilon X \to \upsilon Y$.

A cardinal κ is *measurable* if it admits a countably complete ultrafilter which is not fixed [4]. The *least measurable* cardinal is denoted by $\mathfrak{m}$ and the same symbol is used to denote the least ordinal and the discrete space of cardinality $\mathfrak{m}$. Note that $\mathfrak{m}$ is also the least measurable cardinal in the more restrictive sense [5, after 2.7] of admitting an $\mathfrak{m}$-complete ultrafilter.

Remark 0.1. The set $\upsilon\mathfrak{m} \setminus \mathfrak{m}$ is nonempty.

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1. $\mathfrak{m}$-LIMITS

In this section we introduce λ-limits ([3, §42], [7]) and prove Proposition 1.2 which states that if G is the fundamental group of a compact space then G admits $\mathfrak{m}$-limits. Additionally such limits commute with homomorphisms induced by maps between compact spaces.

Lemma 1.1. *If Y is a locally compact, realcompact space of nonmeasurable cardinality, then $\upsilon(Y \times X) = Y \times \upsilon X$ for all X.*

Proof. See [1, Corollary 2.2]. □

In this paper, Lemma 1.1 is applied for Y compact.

Let λ be an ordinal number. We say that a λ-*limit* is defined on a group G if to each λ-sequence (a_τ) in G there is assigned an element $a \in G$, denoted as

$$a = \lim a_\tau$$

subject to the following postulates:

(i) $\lim(a_\tau \cdot b_\tau) = \lim a_\tau \cdot \lim b_\tau$
(ii) $\lim a_\tau = a$ if $a_\tau = a$ for all $\tau < \lambda$
(iii) $\lim a_\tau = \lim b_\tau$ if $a_\tau = b_\tau$ for $\tau > \tau_0$ with some fixed $\tau_0 < \lambda$.

Note that every λ-limit is a group homomorphism $\lim : G^\lambda \to G$. Our definition of a λ-limit is a direct generalization to nonabelian groups of the definition found in [3, §42] and [7].

Proposition 1.2. *For every point $s \in \upsilon\mathfrak{m} \setminus \mathfrak{m}$ there is an $\mathfrak{m}$-limit $\lim_s$ defined on fundamental groups of compact spaces.*

Proof. Let K be a compact space. Given an $\mathfrak{m}$-sequence (a_τ) in $\pi_1 K$ we choose representatives $S^1 \times \{\tau\} = S^1 \to K$ of a_τ for $\tau < \mathfrak{m}$ and obtain a map $a : S^1 \times \mathfrak{m} \to K$. It induces a map $\upsilon(S^1 \times \mathfrak{m}) \to K$ which, by Lemma 1.1, is a map $S^1 \times \upsilon\mathfrak{m} \to K$. We restrict it to obtain a map

$$\alpha : S^1 = S^1 \times \{s\} \to K$$

which represents an element of $\pi_1 K$. Two different representatives of the elements a_τ are connected by based homotopies, that is, maps $S^1 \times I/\{*\} \times I \to K$. Again by Lemma 1.1 and compactness of $S^1 \times I/\{*\} \times I$, these maps produce a based homotopy between the respective α's. Hence we have a well defined map $\lim_s$ which sends a sequence (a_τ) to an element of $\pi_1 K$ represented by α.

Verification of Properties (i)-(iii) is straightforward. □

Remark 1.3. The limits described in Proposition 1.2 commute with homomorphisms induced by continuous maps, that is, if $f : K \to L$ is a map between compact spaces then

$$\lim_s f_\#(a_\tau) = f_\#(\lim_s a_\tau)$$

for each $\mathfrak{m}$-sequence (a_τ) in $\pi_1 K$.

2. Equal $\mathfrak{m}$-limits

In this section we prove (Proposition 2.2) that the fundamental group of a compact space which has a measurable cardinality admits nontrivial instances of equal $\mathfrak{m}$-limits.

Lemma 2.1. *Let D be a discrete space. For any point $s_0 \in \upsilon D \setminus D$ there is a discrete space X and functions $a, b : X \to D$ and an $s \in \upsilon X \setminus X$ such that $a(x) \neq b(x)$ for each $x \in X$ but $\overline{a}(s) = \overline{b}(s) = s_0$.*

Proof. Let $\Delta = \{(d,d) \mid d \in D\}$ and $X = D \times D \setminus \Delta$. Since the closure of $\{d\} \times (D \setminus \{d\})$ in $\upsilon D \times \upsilon D$ is $\{d\} \times (\upsilon D \setminus \{d\})$ we see that the closure of X is $\upsilon D \times \upsilon D \setminus \Delta$, in particular for each $s_0 \in \upsilon D \setminus D$ the point (s_0, s_0) is in the closure of X. We can define the functions a and b as the inclusion $X \to D \times D$ composed with the standard projections. □

Proposition 2.2. *If K is a compact space and the cardinality of $G = \pi_1 K$ is measurable then there exist $\mathfrak{m}$-sequences (a_τ) and (b_τ) in G and an $s \in \upsilon\mathfrak{m} \setminus \mathfrak{m}$ such that $a_\tau \neq b_\tau$ for all $\tau < \mathfrak{m}$ but $\lim_s a_\tau = \lim_s b_\tau$.*

Additionally we may fix any two distinct elements c and d in G and require that for each $\tau < \mathfrak{m}$ the sets $\{a_\tau, b_\tau\}$ and $\{c, d\}$ are disjoint.

Proof. Let $D \subseteq G$ be a subset of cardinality $\mathfrak{m}$, disjoint from $\{c, d\}$. We represent the elements of D by a map $S^1 \times D \to K$ where D is treated as a discrete space. This leads to the following sequence of maps.

$$\text{(2.3)}\quad S^1 \times \mathfrak{m} \to S^1 \times \upsilon(D \times D) \to S^1 \times \upsilon D \times \upsilon D \to S^1 \times S^1 \times \upsilon D \times \upsilon D = \\ = \upsilon(S^1 \times D) \times \upsilon(S^1 \times D) \to K \times K$$

The first map is the identity on S^1 times the Hewitt realcompactification applied to the map in Lemma 2.1 with $X = \mathfrak{m}$. The second map is induced by a realcompactification of the inclusion $D \times D \to \upsilon D \times \upsilon D$ while the fourth one by the diagonal $S^1 \to S^1 \times S^1$. The equality is induced by the homeomorphism described in Lemma 1.1. The last map is a product of two copies of a realcompactification of the map $S^1 \times D \to K$.

We choose an $s \in \upsilon\mathfrak{m} \setminus \mathfrak{m}$ as in Lemma 2.1. The composition (2.3) restricted to $S^1 \times \{\tau\}$ represents a pair (a_τ, b_τ) for some distinct a_τ and b_τ in G. Restriction of (2.3) to $S^1 \times \{s\}$ represents $(\lim_s a_\tau, \lim_s b_\tau)$. Since the image of s in $\upsilon D \times \upsilon D$ is (s_0, s_0) we see that $\lim_s a_\tau = \lim_s b_\tau$. □

3. Examples

In this section we construct groups of measurable cardinality which cannot be realized as fundamental groups of compact spaces. Proposition 3.2 describes the abelian case while Proposition 3.1 gives a somewhat stronger result in the nonabelian case.

Let $V = \mathbb{F}_2[\mathfrak{m}]$ be an $\mathbb{F}_2$ vector space whose basis is $\mathfrak{m}$. Let $G(V)$ be the subgroup of the automorphism group of V generated by automorphisms induced by those permutations of $\mathfrak{m}$ which fix all but finitely many elements of $\mathfrak{m}$. Let G be a semidirect product $V \rtimes G(V)$. Note that the cardinality of G is $\mathfrak{m}$.

Proposition 3.1. *If K is a compact space then G is not isomorphic to a subgroup of $\pi_1 K$.*

Proof. Suppose to the contrary that we have $G \subseteq \pi_1 K$, up to isomorphism. We fix two constant $\mathfrak{m}$-sequences (c) and (d) in $\mathfrak{m}$ such that $c \neq d$. By Proposition 2.2 we have two $\mathfrak{m}$-sequences (a_τ) and (b_τ) in $\mathfrak{m}$ and an $s \in \upsilon\mathfrak{m} \setminus \mathfrak{m}$ such that for each $\tau < \mathfrak{m}$ the elements a_τ, b_τ, c and d are pairwise distinct but $\lim_s a_\tau = \lim_s b_\tau$. Let $g_\tau \in G(V) \subseteq G$ be those elements which induce by conjugation a cyclic permutation of (a_τ, b_τ, c, d). If $g = \lim_s g_\tau$ then Properties (i) and (ii) of $\lim_s$ imply that g induces a cyclic permutation of $(\lim_s a_\tau, \lim_s b_\tau, c, d)$. Since $\lim_s a_\tau = \lim_s b_\tau$ this is possible only when all the elements of this quadruple are equal. Since $c \neq d$ we have a contradiction. □

The group G in Proposition 3.1 cannot be abelian since any abelian group can be embedded in a product of copies of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$. Since the cardinalities of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are nonmeasurable Corollary 4 in [6] implies that they are the fundamental groups of some compact spaces, hence so are their products. Still, we have the following.

Proposition 3.2. *There exists an abelian group A which is not isomorphic to $\pi_1 K$ for any compact K.*

Proof. Proposition 3.1 in [8] says that there exists an abelian group containing a subgroup A which is $\mathfrak{m}$-pure but not $\mathfrak{m}^+$-pure ($\mathfrak{m}^+$ denotes the successor cardinal of $\mathfrak{m}$).

Suppose that $A = \pi_1 K$ for some compact space K. By Proposition 1.2 we have an $\mathfrak{m}$-limit defined on A. Proposition 5.4 in [8] says that if A admits $\mathfrak{m}$-limits then A is $\mathfrak{m}^+$-pure in any group containing it as an $\mathfrak{m}$-pure subgroup. We have obtained a contradiction. □

As an immediate corollary of Proposition 3.1 or Proposition 3.2 we obtain:

Theorem 3.3. *The following statements are equivalent:*

(i) *There exists a measurable cardinal.*
(ii) *There exists a group which cannot be realized as a fundamental group of a compact. space*

The following class of groups has been communicated to the author by Eda [2]. Let $G = \langle I \mid \mathcal{R}\rangle$ be a group with generators I and relations $\mathcal{R}$. Each element of $\mathcal{R}$ is a finite word on letters i and i^{-1} where $i \in I$. Suppose that

(*) For each $i \in I$ the cardinality of a subset

$$\mathcal{R}_i = \{R \in \mathcal{R} \mid i \text{ or } i^{-1} \text{ appears in } R\}$$

is nonmeasurable.

Proposition 3.4. *The group G as above is a fundamental group of a compact space.*

Proof. Let $\sim$ be the least equivalence relation on I such that $i \sim k$ if there is a word $R \in \mathcal{R}$ which contains i or i^{-1} and k or k^{-1}. Condition (*) implies that the equivalence classes of $\sim$ are nonmeasurable hence G is a free product of nonmeasurable groups G_j where j runs through the set J of the equivalence classes of the relation $\sim$.

By Corollary 4 in [6] there are compact spaces K_j such that $G_j = \pi_1 K_j$. Fix a cardinal κ and embeddings $f_j : K_j \to I^\kappa$ such that each f_j takes the base point to the constant sequence $(\frac{1}{2})$ and its range is contained in $[\frac{1}{2}, 1]^\kappa$. Let $J^\bullet = J \cup \{\infty\}$ be the one point compactification of J considered as a discrete space. A subspace of $I^\kappa \times J^\bullet / \{(0)\} \times J^\bullet$ which is the union of $I^\kappa \times \{\infty\}$ and the images of the maps f_j and the intervals $[(0), (\frac{1}{2})]$ is compact and its fundamental group is a free product of the groups G_j hence is isomorphic to G. $\square$

4. Path connected compact spaces.

Lemma 4.1. *If X is a path connected paracompact space of nonmeasurable cardinality then the path components of βX are of the form X and $\{x\}$ for $x \in \beta X \setminus X$.*

Proof. Theorem 3 in [6] states that X is a path component of βX hence if a path $\alpha : I \to \beta X$ is such that $\alpha(0) \in \beta X \setminus X$ then the whole image of α is contained in $\beta X \setminus X$. By Theorem 2 in [6] this path has to be constant. $\square$

Theorem 4.2. *Any group G of nonmeasurable cardinality is the fundamental group of a path connected compact space Z.*

Proof. Let K be a CW-complex of nonmeasurable cardinality whose fundamental group is G. Let $K_0 = \beta K$ and $f_0 : D \to K_0$ be a map from a discrete space D to K_0 such that for each path component P of K_0 there is exactly one $d \in D$ with $f_0(d) \in P$. Let $C(f_0)$ be the mapping cone of f_0. We define $K_1 = \beta C(f_0)$. We repeat this process inductively and obtain a sequence K_n, $n = 0, 1, 2, \ldots,$ of compact spaces and inclusions $i_n : K_n \hookrightarrow K_{n+1}$.

By Lemma 4.1 we see that $G = \pi_1 K_0$ and $G = \pi_1 C(f_0)$. Since K_0 is a compact C^*-embedded subspace of $C(f_0)$ we see that $\beta C(f_0)/K_0$ is homeomorphic to $\beta(C(f_0)/K_0)$. Since $C(f_0)/K_0$ is a path connected paracompact space Lemma 4.1 implies that for each $x \in K_1 \setminus C(f_0)$ the path component of x in K_1 is $\{x\}$ hence $\pi_1 K_1 = \pi_1 C(f_0)$. By repeating the above argument for each $n = 0, 1, 2, \ldots$ we see that the inclusions $i_n : K_n \to K_{n+1}$ induce isomorphisms of the fundamental groups and $i_n(K_n)$ is contained in a path component of K_{n+1}.

Let T be the telescope of the chain of maps i_n, that is, the space

$$(\coprod_{n=0}^{\infty} K_n \times I)/(x_n, 1) \sim (i_n(x_n), 0)$$

where x_n runs over points in K_n. The telescope T is path connected and its fundamental group is the colimit of $\pi_1 K_n$ which is G. Since T is locally compact we can take its one-point compactification $T^\bullet = T \cup \{\infty\}$. Let $i : T \hookrightarrow T^\bullet$ be the inclusion. Let $p : T \to [0, \infty)$ be a map which sends $(x_n, t) \in K_n \times I$ to $n + t$. Let $g : T \to I$ be defined as $g(x) = \sin^2 p(x)$. Let $\overline{T}$ be the closure of the image of the map $i \times g : T \to T^\bullet \times I$. Since for any positive integer n the space $p^{-1}([0, n])$ is compact we see that $\overline{T}$ has exactly two path connected components: $p^{-1}([0, \infty))$ which is homeomorphic to T and the interval at infinity $\{\infty\} \times I$. The mapping cone of $h : \{0, 1\} \to \overline{T}$ which sends 0 to T and 1 to the interval at infinity is compact path connected and has the fundamental group isomorphic to G. □

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