CHAINS OF GROUP LOCALIZATIONS

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Abstract. We construct long sequences of localization functors L_α in the category of abelian groups such that $L_\alpha \geq L_\beta$ for infinite cardinals $\alpha < \beta$ less than some κ. For sufficiently large free abelian groups F and $\alpha < \beta$ we have proper inclusions $L_\alpha F \subsetneq L_\beta F$.

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We reveal deeper categorical consequences of the proof of [4, Theorem 2.1] than those stated in the original paper. We show that:

($\circ$) There exists a sequence of localization functors $L_\lambda : \mathcal{A}b \to \mathcal{A}b$ in the category of abelian groups, indexed by infinite cardinals λ less than some nonmeasurable cardinal κ, such that if F is a free abelian group of rank at least κ then for $\alpha < \beta$ we have a proper inclusion $L_\alpha F \subsetneq L_\beta F$ which is a localization. More, we have $L_\alpha \geq L_\beta$ for $\alpha < \beta$ and localizations of the integers $R = L_\lambda \mathbb{Z}$ do not depend on λ.

Constructions of this kind have been investigated before. Consider the following sentence:

($*$) There exists a sequence of localization functors $L_\lambda : \mathcal{C} \to \mathcal{C}$ in a category $\mathcal{C}$ and an object F in $\mathcal{C}$ such that for $\alpha < \beta$ we have a proper inclusion $L_\alpha F \subsetneq L_\beta F$ which is a localization.

The statement ($*$) holds in the category of graphs for λ ranging over cardinals less than any κ since the ordered set $[0, \kappa)$, considered as a category, fully embeds into the category of graphs. The validity of ($*$) for λ ranging over all cardinals is equivalent to the negation of Vopěnka's principle – see [1, Lemma 6.3]. In [6] one constructs a functor from the category of graphs to the category of groups which preserves orthogonality between morphisms and objects (see definitions below) – this implies that our remarks on ($*$) hold in the category of groups. Existence of an analogous functor into the category of abelian groups

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(which is conjectured in [6]) would translate the above to the category of abelian groups.

We work in the category of abelian groups $\mathcal{A}b$, although many definitions and properties hold in more general categories (see [2]). *Localization* is a functor $L : \mathcal{A}b \to \mathcal{A}b$ with a natural transformation $a : Id \to \mathcal{A}b$ such that for every $X \in \mathcal{A}b$ we have $a_{LX} = La_X$ and $a_{LX} : LX \to LLX$ is an isomorphism. If a_X is an isomorphism then X is called *L-local*; if Lf is an isomorphism then f is called an *L-equivalence*.

A homomorphism $f : X \to Y$ is *orthogonal* to B (we write $f \perp B$) if f induces, via composition, a bijection $f^* : \mathrm{Hom}(Y, B) \to \mathrm{Hom}(X, B)$. If $f : X \to Y$ is an L-equivalence and B is L-local then $f \perp B$. Conversely, if $f \perp B$ for all L-local B then f is an L-equivalence, and if $f \perp B$ for all L-equivalences f then B is L-local. This implies that the class of L-local groups is closed under limits and retracts, and the class of L-equivalences is closed under colimits – see [2, Proposition 1.3].

For any homomorphism $f : A \to B$ there exists a localization L_f, called an *f-localization*, such that the class of L_f-local groups is $\mathcal{D} = f^\perp = \{D \mid f \perp D\}$, and (it follows that) the class of L_f-equivalences is $\mathcal{E} = \mathcal{D}^\perp = \{g : X \to Y \mid g \perp D \text{ for every } D \in \mathcal{D}\}$. If $f \perp B$ then $a_A = f$ and $B = L_f A$, and it is customary to call such a homomorphism f a *localization*.

For any group B there exists a localization functor L_B, called a *localization at B*, such that the class of L_B-equivalences is $\mathcal{E} = B^\perp = \{g : X \to Y \mid g \perp B\}$ and the class of L_B-local groups is $\mathcal{D} = \mathcal{E}^\perp$. The existence of f-localizations and localizations at a group is proved in [3, Theorem 1].

The class of localizations admits a partial ordering. We say that $L_1 \geq L_2$ if one of the following, equivalent conditions holds:

(1) L_2 factors (uniquely) through L_1.
(2) $L_2 = L_2L_1$.
(3) The class of L_1-local groups contains the class of L_2-local groups.
(4) The class of L_2-equivalences contains the class of L_1-equivalences.

An f-localization is the largest localization among those L for which f is an L-equivalence, while localization at B is the least one among those L for which B is L-local.

If $\kappa \geq \lambda$ are infinite cardinals then by $D^\kappa_{<\lambda}$ we denote the subgroup of $\prod_\kappa D$ consisting of those functions whose support is less than λ.

Lemma 1. *Fix an infinite cardinal λ. If $D^\kappa_{<\lambda}$ is L-local for some $\kappa \geq \lambda$ then $D^\alpha_{<\lambda}$ is L-local for all $\alpha \geq \lambda$.*

Proof. $D^{\lambda}_{<\lambda}$ is a retract of $D^{\kappa}_{<\lambda}$, hence it is L-local. Let $\alpha \geq \lambda$. Each $X \subseteq \alpha$ of cardinality λ induces a projection $\prod_\alpha D \to \prod_X D$. Denoting its image by D_X we obtain $D^{\alpha}_{<\lambda} \to D_X \cong D^{\lambda}_{<\lambda}$. Then $D^{\alpha}_{<\lambda} = \lim_{\substack{X\subseteq\alpha \\ |X|=\lambda}} D_X$ is L-local as a limit of L-local groups. □

Corollary 2. *If $S = \bigoplus_\kappa D$ is L-local for some infinite κ then it is L-local for all κ.*

Lemma 3. *Let $f : A \to B$ be a homomorphism and κ be an infinite regular cardinal greater than the number of generators of A. If D is L_f-local then $D^{\kappa}_{<\kappa}$ is L_f-local.*

Proof. A homomorphism $g : A \to D^{\kappa}_{<\kappa}$ uniquely factors as $A \xrightarrow{f} B \to \prod_\kappa D$, since the product is L_f-local. The union of the supports of all elements in $g(A)$ forms a set X whose cardinality is less than κ; hence $g(A)$ is contained in a subgroup of $D^{\kappa}_{<\kappa}$ isomorphic to $\prod_X D$, hence L_f-local, and therefore g uniquely factors through f. □

Let L be a localization. We look at the composition

$$F_\kappa = \bigoplus_\kappa \mathbb{Z} \xrightarrow{\bigoplus_\kappa a_\mathbb{Z}} \bigoplus_\kappa L\mathbb{Z} \subseteq \prod_\kappa L\mathbb{Z}.$$

Since the product is L-local, it factors as

$$F_\kappa \xrightarrow{a} LF_\kappa \xrightarrow{g} \prod_\kappa L\mathbb{Z} \tag{4}$$

where $a = a_{F_\kappa}$.

Remark 5. Let N^κ_L denote the image of g. Since F_κ is a free group, it is easy to see that N^κ_L is L_a-local. In fact, N^κ_L may be described as the least L_a-local subgroup of $\prod_\kappa L\mathbb{Z}$ which contains $\bigoplus_\kappa \mathbb{Z}$.

Definition 6. Define $\mathsf{support}_\kappa L$ as the least cardinal greater than the cardinalities of the supports of all elements in N^κ_L.

Remark 7. The number $\mathsf{support}_\kappa L$ does not depend on the choice of basis for F_κ: if B and C are two such bases then a bijection $\alpha : B \to C$ induces a diagram

$$\begin{array}{ccccc}
\bigoplus_{b\in B} \mathbb{Z} & \longrightarrow & LF_\kappa & \xrightarrow{g} & \prod_{b\in B} L\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{c\in C} \mathbb{Z} & \longrightarrow & LF_\kappa & \xrightarrow{g'} & \prod_{c\in C} L\mathbb{Z}
\end{array}$$

where the rightmost vertical arrow permutes the components preserving supports of elements.

Definition 8. Define $\mathsf{support}\, L$ to be the supremum of $\mathsf{support}_\kappa L$ over all cardinals κ, or ∞ if this class of cardinals is unbounded.

An embedding of a subset $X \subseteq \kappa$ induces a diagram

$$\begin{array}{ccccc} \bigoplus_\kappa \mathbb{Z} & \longrightarrow & LF_\kappa & \xrightarrow{g} & \prod_\kappa L\mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_X \mathbb{Z} & \longrightarrow & LF_X & \xrightarrow{g'} & \prod_X L\mathbb{Z} \end{array}$$

where the vertical arrows are retractions. This allows comparing possible cardinalities of supports of elements of $N_L^\kappa \subseteq \prod_\kappa L\mathbb{Z}$ for different κ's, and therefore it proves:

Lemma 9. *If $\mathsf{support}_\kappa L \leq \kappa$ then $\mathsf{support}\, L = \mathsf{support}_\kappa L$.*

Lemma 10. *Let L be a localization. The following are equivalent:*

(1) *$\mathsf{support}\, L = \omega_0$.*
(2) *$LF_{\omega_0} = \bigoplus_{\omega_0} L\mathbb{Z}$.*
(3) *For any κ we have $LF_\kappa = \bigoplus_\kappa L\mathbb{Z}$.*

Proof. (3) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (2) are obvious; (2) $\Longrightarrow$ (3) follows from Corollary 2. It remains to prove (1) $\Longrightarrow$ (3). If $\mathsf{support}\, L = \omega_0$ then we have an epimorphism $g : LF_\kappa \to N_L^\kappa \cong \bigoplus_\kappa L\mathbb{Z}$. Since LF_κ is L-local and the target of g is L-equivalent to the free group F_κ via an L-equivalence $\bigoplus_\kappa(\mathbb{Z} \to L\mathbb{Z})$ we see that g has a right inverse r. Then $r(N_L^\kappa)$ is a retract of LF_κ which contains F_κ, thus r is onto and g is an isomorphism as claimed. □

A localization satisfying the conditions of Lemma 10 is called in [4] a *standard localization*.

Lemma 11. *Let κ be an infinite cardinal less than the first measurable cardinal. Then there exists a localization L such that $\mathsf{support}\, L > \kappa$.*

Proof. At the heart of the proof of [4, Theorem 2.1] lies a construction of a localization homomorphism $\varepsilon : F_\kappa \to M$ such that for a certain group R we have $\bigoplus_\kappa R \subseteq M \subseteq \prod_\kappa R$ and M contains functions which are nowhere zero and $R = L_\varepsilon \mathbb{Z}$. This implies our claim. □

Theorem 12. *Let κ be an infinite cardinal less than the first measurable cardinal. There exists a sequence of localization functors L_α for $\alpha < \kappa$, such that:*

(a) *$\mathsf{support}\, L_\alpha = \alpha^+$,*
(b) *$L_\alpha \geq L_\beta$ for $\alpha < \beta < \kappa$,*
(c) *$L_\alpha F_\kappa \subsetneq L_\beta F_\kappa$ for $\alpha < \beta < \kappa$,*

where α^+ is the successor cardinal of α.

Proof. Let L be the localization from Lemma 11 and $f_\alpha : F_\alpha \to LF_\alpha$ be the localization homomorphism. Define $L_\alpha = L_{f_\alpha}$. Since $L_\alpha F_\alpha = LF_\alpha$ is a retract of LF_κ, an argument as in the proof of Lemma 9 implies that $\mathsf{support}\, L_\alpha > \alpha$. Lemma 3 for $\kappa = \alpha^+$ and Lemma 1 imply that $R^\kappa_{<\alpha^+}$ is L_α-local for all $\kappa > \alpha$, hence Remark 5 implies that $\mathsf{support}\, L_\alpha \leq \alpha^+$, which yields (a). Since for $\alpha < \beta$ the map f_α is a retract of f_β, items (b) and (c) follow easily. □

If $f : \mathbb{Z} \to R = L_\varepsilon \mathbb{Z}$ is an L_ε localization of $\mathbb{Z}$ as in the proof of Lemma 11 then the f-localization L_f is strictly greater, while the localization at R, L_R, is strictly less than all the localizations L_α. We do not know if $L = L_R$; it is still conceivable that $\mathsf{support}\, L_R$ might exceed κ^+.

In the proof of Lemma 11 the groups R and $M = LF_\kappa$ have the same cardinality $\lambda \geq 2^\kappa$, hence also the groups $L_\alpha F_\kappa$ have cardinality λ each. This cannot happen if we want α to run over all cardinals, as we speculated in the introduction.

In principle, one could construct similar sequences of localizations based on the structure of the kernels of maps g in Diagram (4), but we are unaware of any examples of nontrivial kernels of g. Dugas and Feigelstock prove in [5, Theorem 1.8] that in certain cases these kernels must be trivial.

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