How comprehensive is the category of abelian groups?

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When a category is comprehensive?

When the category of graphs fully embeds into it.

Why graphs?

- Every accessible category admits a full embedding into graphs (Adámek-Rosický, 1994)
- Assuming that the measurable cardinals are bounded above every concrete category fully embeds into graphs (Hedrlín-Kučera, 1969 unpublished)

Some categories admit a full embedding of $\mathcal{G}$*raphs*:

- Category of semigroups (Hedrlín-Lambek, 1969)
- Category of integral domains (Fried-Sichler, 1977)

Some more categories admit an "almost" full embedding of $\mathcal{G}$*raphs*:

- Category of metric spaces (up to constant maps – Trnková, 1972)
- Category of paracompact spaces (up to constant maps – Koubek, 1974)
- Category of groups (up to trivial homomorphisms and conjugation in the targets)
- The unpointed homotopy category (up to null-homotopic maps)
- Category of abelian groups (up to ???)

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$$G : \mathcal{G}raphs \to \mathcal{A}b$$

- full embedding:

$$\mathrm{Hom}(X, Y) \xrightarrow{\cong} \mathrm{Hom}(GX, GY)$$

- "almost" full for abelian groups:

$$\mathbb{Z}[\mathrm{Hom}_{\mathcal{G}raphs}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}b}(GX, GY)$$

where $\mathbb{Z}[S]$ is the free group with basis S.

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Natural completion of abelian groups $\eta_A : A \to \widehat{A}$.

- $\widehat{A} \cong \lim_{n\in\mathbb{N}} A/nA$
- $\widehat{A} \cong \prod_p A_p^\wedge$
- $\ker \eta_A$ is the divisible subgroup of A.

The completion is an idempotent functor (localization, reflector), that is:

- $\widehat{\eta}_A : \widehat{A} \xrightarrow{\cong} \widehat{\widehat{A}}$
- For all groups A, B the map

$$\mathrm{Hom}(\widehat{A}, \widehat{B}) \xrightarrow{\eta_A} \mathrm{Hom}(A, \widehat{B})$$

  is an isomorphism.

If $A \subseteq C \subseteq \widehat{A}$ and C is pure in $\widehat{A}$ then $\widehat{C} = \widehat{A}$. In particular the inclusion $A \subseteq C$ induces an isomorphism

$$\mathrm{Hom}(A, \widehat{A}) \cong \mathrm{Hom}(C, \widehat{A})$$

Theorem (Corner, 1963)

Let A be a ring of cardinality at most continuum, whose additive group is free. Then there exists a group $\mathbb{A}$ such that

(a) $A \subseteq \mathbb{A} \subseteq \widehat{A}$ *as left A-modules.*

(b) $A \cong \mathrm{Hom}(\mathbb{A}, \mathbb{A})$

(c) $|A| = |\mathbb{A}|$

The construction:

Choose elements α_a, β_a ($a \in A$) of $\widehat{\mathbb{Z}}$ that are algebraically independent over $\mathbb{Z}$. Define elements e_a ($a \in A$) of $\widehat{A}$ as

$$e_a = \alpha_a \cdot 1 + \beta_a \cdot a$$

and take $\mathbb{A}$ to be the pure subgroup of $\widehat{A}$ generated by A and Ae_a ($a \in A$).

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Construction of the functor G

Let Γ be a full subcategory of $\mathcal{G}raphs$ whose objects are representatives of countable graphs.
Let $A = \mathbb{Z}[\Gamma]$ be the ring whose additive group is free with the basis consisting of the identity 1 and the maps $\varphi : X \to Y$ in Γ.
If $\psi : M \to N$ is another map in Γ then

- the product $\varphi\psi$ in A is the composition $\varphi\psi$ if $N = X$
- $\varphi\psi = 0$ if $N \neq X$

$A \cong \mathrm{Hom}(\mathbb{A}, \mathbb{A})$
The identity $\mathrm{id}_X : X \to X$ in Γ is an idempotent of A hence we have $\mathbb{A} \cong \mathrm{id}_X \cdot \mathbb{A} \oplus (1 - \mathrm{id}_X) \cdot \mathbb{A}$.

Define:

- On Γ: $GX = \mathrm{id}_X \cdot \mathbb{A}$, $G\varphi =$ left multiplication by φ.
- Extend G to all countable graphs.
- For arbitrary graph X, let $[X]$ be the poset of countable subgraphs $C \subseteq X$, define:

$$GX = \operatorname*{colim}_{C \in [X]} GC$$

Proof of $\mathbb{Z}[\mathrm{Hom}_{\mathcal{G}raphs}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}b}(GX, GY)$

Thanks to the A-module inclusions $A \subseteq \mathbb{A} \subseteq \widehat{A}$, the proof is almost tautological.

P1 Let Γ_X denote the set of maps $C \to X$ in Γ. Left multiplication by id_X is the identity on Γ_X and zero on $1 - \mathrm{id}_X$ and $\Gamma \setminus \Gamma_X$. We obtain:

$$\langle \Gamma_X \rangle \subseteq GX \subseteq \widehat{\langle \Gamma_X \rangle}$$

P2 Every $u \in GX$ may be uniquely written as the finite sum

$$u = \sum z_i \sigma_i$$

where $\sigma_i \in \Gamma_X$ and $z_i \in \widehat{\mathbb{Z}}$.

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P3 If $\varphi : X \to Y$ is one-to-one then so is $G\varphi : GX \to GY$.

$\varphi : \widehat{\langle \Gamma_X \rangle} \to \widehat{\langle \Gamma_Y \rangle}$ is a monomorphism
directed colimits preserve monomorphisms

P4 Every homomorphism $h : GX \to GY$ with X and Y in Γ is uniquely represented as left multiplication by an $a = \sum k_i \sigma_i$ in $\mathbb{A}$ where k_i are nonzero integers and $\sigma_i : X \to Y$ are distinct maps in Γ. That is in Γ we have:

$$\mathbb{Z}[\mathrm{Hom}(X, Y)] \cong \mathrm{Hom}(GX, GY)$$

The composition $\mathbb{A} \xrightarrow{\mathrm{id}_X \cdot \_} GX \xrightarrow{h} GY \subseteq \mathbb{A}$ is represented this way.

P5 For X, Y in Γ and any $h : GX \to GY$ if $h(\mathrm{id}_X) = 0$ then $h = 0$.

$h(x) = ax$, a as above, and $h(\mathrm{id}_X) = a$.

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P6 Let W and $\varphi : X \to Y$ be in Γ.
If

$$\begin{array}{ccc} \{\mathrm{id}_W\} & \xrightarrow{\tilde{h}} & GX \\ \cap\!\!\!\mid & \nearrow & \downarrow G\varphi \\ GW & \xrightarrow{h} & GY \end{array}$$

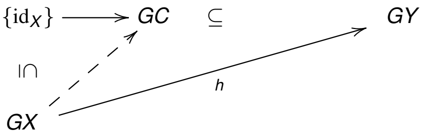
Then the dashed arrow exist.

$$h(\mathrm{id}_W) = G\varphi u, \quad u = \sum z_j \tau_j$$

$$\sum k_i \sigma_i = \sum z_j \varphi \tau_j$$

P7 If X is countable then for any homomorphism $h : GX \to GY$ there exists a countable subgraph $C \subseteq Y$ such that h factors through $GC \subseteq GY$.

$GX = \mathrm{colim}_{C \in [X]} GC$ is a directed colimit $\implies \mathrm{id}_X \in GC$ for some countable C.



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P5 $\implies$ uniqueness

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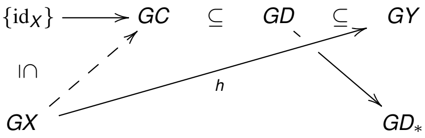

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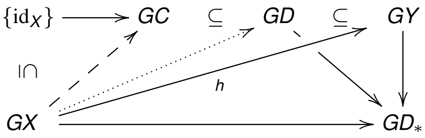

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P8 If X is countable then

$$\mathbb{Z}[\mathrm{Hom}(X, Y)] \cong \mathrm{Hom}(GX, GY)$$

is an isomorphism.

P4 $\Longrightarrow$ isomorphism when Y is also countable
P7 and directedness of the colimit $\Longrightarrow$ the isomorphism

$$\mathbb{Z}[\mathrm{Hom}(X, Y)] \cong \mathbb{Z}[\mathrm{Hom}(X, \operatorname*{colim}_{C \in [Y]} C)] \cong \operatorname*{colim}_{C \in [Y]} \mathbb{Z}[\mathrm{Hom}(X, C)] \underset{\substack{\Uparrow \\ P4}}{\xrightarrow{\cong}}$$

$$\to \operatorname*{colim}_{C \in [Y]} \mathrm{Hom}(GX, GC) \underset{\substack{\Uparrow \\ P7}}{\xrightarrow{\cong}} \mathrm{Hom}(GX, \operatorname*{colim}_{C \in [Y]} GC) \cong \mathrm{Hom}(GX, GY).$$

P9 Let $\{S_i\}_{i \in I}$ be a diagram of sets.
We obtain $\lambda : \mathbb{Z}[\lim S_i] \to \lim \mathbb{Z}[S_i]$

- If I is codirected then λ is one-to-one.
- If I is countably codirected then λ is an isomorphism.

Theorem

For any graphs X and Y

$$\mathbb{Z}[\mathrm{Hom}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}(GX, GY)$$

Proof.

$$\mathbb{Z}[\mathrm{Hom}(X, Y)] \cong \mathbb{Z}[\mathrm{Hom}(\operatorname*{colim}_{C\in[X]} C, Y)] \cong \mathbb{Z}[\lim_{C\in[X]} \mathrm{Hom}(C, Y)] \xrightarrow{\lambda}$$

$$\to \lim_{C\in[X]} \mathbb{Z}[\mathrm{Hom}(C, Y)] \xrightarrow{\lim \gamma} \lim_{C\in[X]} \mathrm{Hom}(GC, GY) \cong$$

$$\cong \mathrm{Hom}(\operatorname*{colim}_{C\in[X]} GC, GY) \cong \mathrm{Hom}(GX, GY).$$

P9 $\implies$ λ is an isomorphism
P8 $\implies$ $\lim \gamma$ is an isomorphism

□

Finite approximation

Let $\Gamma_{fin} \subseteq \Gamma$ category of finite graphs
$A_{fin} = \mathbb{Z}[\Gamma_{fin}] = \mathrm{Hom}(\mathbb{A}_{fin}, \mathbb{A}_{fin}), \quad G_{fin}X = \mathrm{colim}_{C \in [X]_{fin}} G_{fin}C$

$$\gamma : \mathbb{Z}[\mathrm{Hom}_{\mathcal{G}raphs}(X, Y)] \to \mathrm{Hom}(G_{fin}X, G_{fin}Y).$$

1. If X is infinite then $|X| = |G_{fin}X|$.
2. The inclusion of A_{fin}-modules $\mathbb{A}_{fin} \subseteq \mathbb{A}$ yields a natural transformation $h : G_{fin} \to G$ consisting of inclusions.
3. γ is one-to-one.
4. If X is finite then γ is an isomorphism.
5. If $\mathrm{Hom}(X, Y)$ is finite then γ is an isomorphism.

Rigid systems of groups

Theorem (Shelah, 1974)

For any infinite cardinal κ there exists a system of groups $\{M_i\}_{i<2^\kappa}$ such that $|M_i| = \kappa$ and if $h : M_i \to M_j$ is a nonzero homomorphism then $i = j$ and h is a multiplication by an integer.

Alternative proof.

Vopěnka and Hedrlín (1965) proved that for any infinite cardinal κ there exists a rigid system of graphs $\{X_i\}_{i<2^\kappa}$, each graph of cardinality κ. Let $M_i = G_{fin}X_i$.

Generalized pure subgroups

If κ is an infinite cardinal, a subgroup N of M is said to be κ-pure if N is a direct summand of every subgroup N' such that $N \subseteq N' \subseteq M$ and $|N'/N| < \kappa$.

Theorem (Megibben, 1972)

For every infinite cardinal κ there exists a group M containing a κ-pure subgroup N which is not κ^+-pure.

Alternative proof.

Let X_α be the graph representing the order relation of the ordinal α. Let $N = G_{fin} X_\kappa$ and $M = G_{fin}(X_\kappa \vee X_{\kappa+1})$.

A class of groups

Theorem

There exists a class of groups M_α indexed by all ordinals α such that for $\alpha < \beta$ we have $\mathrm{Hom}(M_\beta, M_\alpha) = 0$.

Proof.

Let X_α be the graph representing the order relation of α.
Let $M_\alpha = GX_\alpha$.

The following are equivalent:

1. Negation of Vopěnka's Principle
2. There exists a rigid class of graphs
3. There exists a rigid class of groups

$1 \iff 2$ Definition

$3 \implies 2$ Adámek-Rosický, 1994

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Chains of group localizations

For any ordinal λ there exists a chain of groups M_α, $\alpha < \lambda$ such that the inclusions $M_\alpha \subseteq M_\beta$ are localizations for $\alpha < \beta < \lambda$. That means they induce isomorphisms

$$\mathrm{Hom}(M_\beta, M_\beta) \cong \mathrm{Hom}(M_\alpha, M_\beta)$$

The following are equivalent:

1. Negation of Vopěnka's Principle
2. $\mathcal{O}rd$ fully embeds into $\mathcal{G}raphs$
3. The chain above may be indexed by all ordinals

$3 \implies 2 \iff 1$ Adámek-Rosický, 1994

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Three definitions of a localization $L : \mathcal{C} \to \mathcal{C}$

1. L is a left adjoint of some inclusion functor $\mathcal{D} \subseteq \mathcal{C}$.
2. L is a functor with coaugmentation $\eta : Id \to L$ such that $\eta_{LX} = L\eta_X : LX \to LLX$ is an isomorphism
3. (HPS)
   - $L\eta : LX \to LLX$ is an isomorphism for all X
   - For all X and Y the morphism η_X induces a bijection

$$\mathrm{Hom}(LX, LY) \to \mathrm{Hom}(X, LY)$$

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Localizations may be viewed as projections
onto the class of local objects $\mathcal{D} = \{Z \mid \eta_Z : Z \xrightarrow{\cong} LZ\}$
along the class of L-equivalences $\mathcal{E} = \{f \mid Lf \text{ is an isomorphism}\}$
For every

$f : A \to B$ in $\mathcal{E}$, an L-equivalence and
Z in $\mathcal{D}$, an L-local object

we have:

$$f^* : \mathrm{Hom}(B, Z) \xrightarrow{\cong} \mathrm{Hom}(A, Z)$$

Orthogonality pairs

We say that $f : A \to B$ is orthogonal to Z, and write $f \perp Z$, if

$$f^* : \mathrm{Hom}(B, Z) \xrightarrow{\cong} \mathrm{Hom}(A, Z)$$

A pair $(\mathcal{E}, \mathcal{D})$ is an orthogonality pair if $\mathcal{E} = \mathcal{D}^\perp$ and $\mathcal{D} = \mathcal{E}^\perp$.
A localization always yields an orthogonality pair.

Whether every orthogonality pair is associated with a localization depends on set theory

in $\mathcal{G}raphs$:

- NO is consistent with ZFC
- weak Vopěnka's principle is equivalent to YES

A category $\mathcal{D}$ is reflective if $(\mathcal{D}^\perp, \mathcal{D})$ is associated with some localization.

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Theorem (Isbell problem, 1966)

The following are equivalent:

1. *every closed under limits full subcategory of $\mathcal{G}$raphs is reflective (weak Vopěnka's Principle)*
2. *every closed under limits full subcategory of $\mathcal{A}b$ is reflective*

$1 \implies 2$ was proved by Adámek and Rosický (1994)
$2 \implies 1$ follows from properties of G

A class $\mathcal{D}$ is called a small-orthogonality class if it is of the form $\mathcal{D} = \{f\}^{\perp}$ for some morphism f.

Theorem

The following are equivalent:

1. *every closed under limits full subcategory of $\mathcal{G}$raphs is a small-orthogonality class (Vopěnka's Principle)*
2. *every closed under limits full subcategory of $\mathcal{A}b$ is a small-orthogonality class*

$1 \implies 2$ was proved by Adámek and Rosický (1994)
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Stable homotopy theory

Definition (Hovey-Palmieri-Strickland)

An exact localization in a stable homotopy category $\mathcal{C}$ is a functor L from $\mathcal{C}$ to itself with a natural transformation $\eta : Id \to L$ such that:

1. $L\eta : LX \to LLX$ is an isomorphism for all X
2. For all X and Y the morphism η_X induces an isomorphism

$$[LX, LY] \to [X, LY]$$

3. (HPS) If $LX = 0$ then $L(X \wedge Y) = 0$ for all Y

If L satisfies 1 – 2 only we call it a localization.

In the homotopy category of spectra $\mathcal{S}$ condition 3 means that L commutes with the suspension and preserves exact triangles.

Let $L : \mathcal{S} \to \mathcal{S}$ be a localization in the homotopy category of spectra. Let $f : A \to B$, Lf be an equivalence and $Z \cong LZ$. The map of the function spectra

$$f^* : F(B, Z) \to F(A, Z)$$

induces isomorphisms of the homotopy groups in the nonnegative degrees. If L is a exact then f^* induces isomorphisms of all the homotopy groups.

The corresponding notions of orthogonality between f and Z lead to orthogonality pairs and exact orthogonality pairs.

Theorem (Casacuberta-Gutiérrez-Rosický, 2011 preprint)

Assuming Vopěnka's Principle every class in $\mathcal{S}$, closed under homotopy limits, is reflective.

Theorem

Assuming negation of weak Vopěnka's Principle there exists an orthogonality pair $(\mathcal{E}, \mathcal{D})$ in $\mathcal{S}$ such that $\mathcal{D}$ is not reflective (it is closed under homotopy limits).

These theorems come close to answering the HPS question about existence of exact localizations.
Note that Vopěnka's and weak Vopěnka's Principles are equivalent to their analogues in many categories.
In homotopy category *Ho*:

- $\mathcal{G}raphs \implies Ho$ (Casacuberta-Scevenels-Smith, 2005)
- $Ho \implies \mathcal{G}raphs$ (AP, 2010)

Theorem

Assuming negation of weak Vopěnka's Principle there exists an orthogonality pair in $\mathcal{S}$ which is not associated with a localization.

Proof. (hint)

Negation of weak Vopěnka's Principle implies existence of an orthogonality pair $(\mathcal{E}, \mathcal{D})$ in $\mathcal{A}b$ which is not associated with any localization.
We have a full embedding $\mathcal{A}b \to \mathcal{S}$, $A \mapsto HA$ where $(HA)_n = K(A, n)$.

$$\pi_n(F(HA, HB)) = [HA, \Sigma^{-n}HB] = \begin{cases} \mathrm{Hom}(A, B) & n = 0 \\ \mathrm{Ext}(A, B) & n = -1 \\ 0 & n > 0 \end{cases}$$

The embedding preserves orthogonality hence $(H\mathcal{E}, H\mathcal{D})$ extends to an orthogonality pair $(H\mathcal{D}^{\perp}, H\mathcal{D}^{\perp\perp})$ in $\mathcal{S}$.

Questions

- Are the following statements equivalent?
  1. weak Vopěnka's Principle
  2. every exact orthogonality pair in $\mathcal{S}$ is associated with a localization
- Does $GX \cong GY$ imply $X \cong Y$?
  - Yes if X or Y is finite
  - $GX \cong GY$ implies $|X| = |Y|$
- Is there an analogue of the functor G such that GX has finite rank for finite X?
- Can we modify the construction of G so that GX is an E-ring for rigid graphs X?
  More generally: is it possible to reprove some results, obtained by the Shelah Black Box, using methods of category theory.

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  - Yes if X or Y is finite
  - $GX \cong GY$ implies $|X| = |Y|$
- Is there an analogue of the functor G such that GX has finite rank for finite X?
- Can we modify the construction of G so that GX is an E-ring for rigid graphs X?
  More generally: is it possible to reprove some results, obtained by the Shelah Black Box, using methods of category theory.

Properties of $G : \mathcal{G}raphs \to \mathcal{A}b$

1. $\mathbb{Z}[\mathrm{Hom}_{\mathcal{G}raphs}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}(GX, GY)$
2. $f \perp Z$ if and only if $Gf \perp GZ$
3. the cardinality of GX is at least the continuum
4. G preserves countably directed colimits
5. G preserves monomorphisms
6. for every $a \in GX$ there exists a countable subgraph $C \subseteq X$ such that $a \in GC$
7. G does <u>not</u> preserve products.