Definition. Direct sum of objects $A, B \in \text{ob} \mathcal{C}$ is an object $C$ together with morphisms $A \xrightarrow{i_A} C$, $B \xrightarrow{i_B} C$ satisfying the following universality condition: for every pair of morphisms $A \xrightarrow{f} U$, $B \xrightarrow{g} U$ there exists the unique morphism $C \xrightarrow{h} U$ such that $f = h \circ i_A$, $g = h \circ i_B$. Direct product of objects $A, B \in \text{ob} \mathcal{C}$ is defined dually (inverting arrows).

Zad. 1. Note that a direct sum of two objects in the category $\mathcal{C}$ is a direct product in the opposite category $\mathcal{C}^{\text{op}}$.

Zad. 2. Describe direct sums and direct products in the categories listed in Series 1. Zad. 1.

Zad. 3. Generalize definition of the direct sum (product) to an arbitrary family of objects $\{A_i\}_{i \in I}$.

Zad. 4. Let $A, B \in \text{ob} (\mathcal{C})$. The object $C$ with morphisms $A \xrightarrow{i_A} C$, $B \xrightarrow{i_B} C$ is a direct sum if and only if the morphisms define and isomorphism of the functor $F_{A,B}(X) := \text{Mor}_{\mathcal{C}}(A, X) \times \text{Mor}_{\mathcal{C}}(B, X)$ with the contravariant functor represented by the object $C$. Conversely, any equivalence of the functors defines a direct sum of the objects $A, B$.

Zad. 5. Consider "natural" examples of inclusions of categories $\mathcal{D} \subset \mathcal{C}$ which do preserve direct sums (resp. products) and such which do not preserve them.

Zad. 6. Let $\mathcal{C}$ be a small category and $\mathcal{F}(\mathcal{C}, \text{Set})$ denote the category of functors $\mathcal{C} \to \text{Set}$ whose objects are functors and morphisms are natural transformation between functors. Prove that in the category $\mathcal{F}(\mathcal{C}, \text{Set})$ there exist direct sums and direct products of any two objects (even if they don’t exist in $\mathcal{C}$!). Does the Yoneda embedding preserve direct sums and products?

Zad. 7. For the selected categories listed in Series 1. Zad. 1 analyze existence of direct sums (products) in the categories of "objects under $\mathcal{C}$” and "objects over $\mathcal{C}$” defined in Series 1. Zad. 4. Does the forgetful functors defined in Series 1. Zad. 4. preserve direct sums?

Zad. 8. If a functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence of categories then it preserves direct sums and direct products.

Zad. 9. Show that a functor $F: \mathcal{C} \to \mathcal{D}$ which admits a right adjoint functor preserves direct sums (but not necessary direct products).

Zad. 10. Give a definition of a monomorphism (resp. epimorphism) in an arbitrary category, such that it coincides with definitions known from algebra. Check whether direct products and direct sums preserve monomorphism (resp. epimorphism).