Series 1: Categories and functors

Zad. 1. Show that if a functor \( F: \mathcal{C} \to \mathcal{D} \) has a left (resp. right) adjoint functor then this adjoint functor is unique up to natural equivalence.

Zad. 2. Let \( F: \mathcal{C} \to \mathcal{D} \) and \( G: \mathcal{D} \to \mathcal{C} \) is a pair of adjoint functors. Show, that there exist natural transformations \( \Phi: FG \to \text{id}_\mathcal{D} \) and \( \Psi: \text{id}_\mathcal{C} \to GF \) such that the triangles of natural transformations:

\[
\begin{array}{ccc}
F & \xrightarrow{id} & F \\
\downarrow F\Phi & & \downarrow \Phi F \\
FGF & \downarrow & \end{array}
\]

and

\[
\begin{array}{ccc}
G & \xrightarrow{id} & G \\
\downarrow \Psi G & & \downarrow G\Phi \\
GFG & \downarrow & \end{array}
\]

commute.

Let \( F: \mathcal{C} \to \mathcal{D} \) and \( G: \mathcal{D} \to \mathcal{C} \) be a pair of functors such that there exist natural transformations \( \Phi: FG \to \text{id}_\mathcal{D} \) and \( \Psi: \text{id}_\mathcal{C} \to GF \) such that the above triangles of natural transformations commute. Show that \( F, G \) is a pair of adjoint functors.

Zad. 3. Show that if \( F: \mathcal{C} \to \mathcal{D} \) is an equivalence of categories then the functor \( G: \mathcal{D} \to \mathcal{C} \) establishing this equivalence is both right and left adjoint to \( F \). Is the converse true?

Series 2: Representability, limits.

Zad. 4. Let \( F: \mathcal{C} \to \mathcal{D} \) be a functor. For an object \( Y \in \text{ob}\mathcal{D} \) consider the functor \( F_Y: \mathcal{C}^{\text{op}} \to \text{Set}, \) \( F_Y(X) = \text{Mor}_{\mathcal{D}}(F(X), Y) \). Prove that if for every object \( Y \in \text{ob}\mathcal{D} \) functor \( F_Y \) is representable then there exists a functor \( G: \mathcal{D} \to \mathcal{C} \) right adjoint to \( F \).

Zad. 5. Let \( \mathcal{I} \) be a small category and \( F: \mathcal{I} \to \mathcal{C} \) a diagram in \( \mathcal{C} \). For every object \( X \in \text{ob}\mathcal{C} \) define a constant functor \( \Delta_X: \mathcal{I} \to \mathcal{C} \), which to every object \( i \in \text{ob}\mathcal{I} \) assigns \( X \) and to every morphism in \( \mathcal{I} \) assigns \( \text{id}_X \). \( \Delta: \mathcal{C} \to \text{Funct}(\mathcal{I}, \mathcal{C}) \) for which \( \Delta(X) = \Delta_X \) and for a morphism \( f: X \to X' \) in \( \mathcal{C} \) is a
natural transformation \( \Delta(f) : \Delta X \to \Delta X' \). \( \Delta(f)(i) = f : \Delta X(i) = X \to \Delta X'(i) = X' \), Check that \( \Delta \) is indeed a functor. Prove that \( \lim_1 F \) exists iff the functor \( \text{Mor}_{\text{Funct}(\mathcal{I}, \mathcal{C})}(\Delta -, F) : \mathcal{C}^{op} \to \text{Set} \) is representable and \( \lim F \) is the representing object.

Formulate the analogous statement for colim.

Zad. 6. Let \( \mathcal{I} \) be a small category and consider a diagram \( F : \mathcal{I} \to \text{Set} \) in the category of sets given by a representable functor \( \text{Mor}_\mathcal{I}(i_0, \cdot) \). Find \( \text{colim} F \). (hint: Yoneda helps a lot!)

1 Series 3: Cofibrations and Fibrations.

Zad. 7. Show that if

\[
\begin{array}{ccc}
A & \xrightarrow{j} & X \\
\downarrow f & & \downarrow h \\
Y & \xrightarrow{g} & W
\end{array}
\]

is a push out diagram in \( \text{Top} \) then for every space \( Z \) the induced diagram

\[
\begin{array}{ccc}
\text{map}(W, Z) & \xrightarrow{g^*} & \text{map}(Y, Z) \\
\downarrow h^* & & \downarrow f^* \\
\text{map}(X, Z) & \xrightarrow{j^*} & \text{map}(A, Z)
\end{array}
\]

is a pull back diagram. (This was the key step in proving that for a map \( f : X \to Y \), \( \text{map}(Z(f), Z) = P(f^*) \).)

Zad. 8. Present the map \( X \sqcup X \to X \) sending each summand identically onto \( X \) as the composition of a cofibration and homotopy equivalence.

Present the diagonal map \( \Delta : Y \to Y \times Y \) as the composition of a homotopy equivalence and a fibration.

Zad. 9. Present the map \( X \to \{\ast\} \) as the composition of a cofibration and homotopy equivalence.

Present the map \( \{\ast\} \to X \) as the composition of a homotopy equivalence and a fibration.