

INTRODUCTION TO SPHERICAL FUNCTORS

AGNIESZKA BODZENTA

ABSTRACT. We recall various definitions of spherical functors, both in the setting of triangulated and pre-triangulated DG categories. We discuss how these definitions are related to each other. Then we discuss examples of triangle spherical functors, i.e. the gluing functors for 4-periodic semi-orthogonal decompositions. We recall two known examples, of an effective Cartier divisor and the flop-flop equivalence. Then we present a new example of a 4-periodic decomposition for the EZ-spherical functor. We conclude with a discussion of possible topics for further research.

CONTENTS

1. Introduction	1
2. Overview	2
2.1. Spherical objects	2
2.2. Spherical functors	3
2.3. Spherical DG functors	5
2.4. Semi-orthogonal decompositions and mutations	6
2.5. Gluing via spherical functors	7
2.6. Triangle spherical functors	7
3. Definitions of spherical functors	8
3.1. Spherical functors between triangulated categories	8
3.2. Examples when the assumptions are satisfied	11
3.3. DG gluing	12
3.4. Spherical functors and 4-periodic decompositions	14
3.5. Any autoequivalence is a spherical twist	16
4. Examples of periodic SODs	16
4.1. Periodic SODs for an effective Cartier divisor	17
4.2. Periodic SODs for flops	19
5. Spherical functor for a blow-up	21
5.1. Periodic decompositions on the quotient category	21
5.2. Periodic decomposition for a blow-up	23
5.3. More examples of spherical functors	24
5.4. Where to go from here	25
References	25

1. INTRODUCTION

These are extended notes for the series of lectures on spherical functors during the VIU Summer School “Perverse Sheaves and their Categorification in Geometry and Representation Theory”. The first aim of these lectures is to gather various definitions of spherical functors discussed in the literature [Ann13, Add16, Plo05, Kuz19, Mea21, AL25] and to understand how they are related one to another. Then examples of spherical functors will be given.

Most definitions of spherical functors require an extra triangulated structure on the space of functors, allowing one to consider cones of adjunction unit and counit. This extra structure can be given either by considering Fourier-Mukai transforms between smooth projective varieties [Ann13, Add16, Plo05, Kuz19, Mea21] or by passing to Frobenius exact categories [Rou04], see

Section 3.2. One might also consider quasi-functors between pre-triangulated DG categories [AL17].

However, one definition does not require any extra assumption. One can define triangle spherical functors [BD24] as the gluing functors for 4-periodic semi-orthogonal decompositions [HLS16, KS14, DKS23]. From the author's point of view this is the most elegant approach and for this reason examples discussed in details in these notes will be examples of periodic decompositions.

The first lecture, planned for 50 minutes, recalls spherical objects and states various definitions of spherical functors. Examples of both concepts are given without much detail. The second lecture, planned for 100 minutes, is the technical one. We shall see, with some proofs, how various definitions are related to each other.

The third and the fourth lecture, both planned for 50 minutes, shall focus mostly on examples. In the third lecture two known examples of periodic decompositions will be discussed. The first one concerns the root stack of an effective Cartier divisor $i: D \rightarrow X$ and proves that $i_*: \mathcal{D}^b(D) \rightarrow \mathcal{D}^b(X)$ is (triangle) spherical [BD24]. The second one concerns the (quotient) of the derived category of the fiber product of two varieties related by a flop. It shows that the flop-flop functor is a twist for an appropriate spherical functor [BB22, Bar20].

In the last lecture a new example of a periodic decomposition will be given. The corresponding triangle spherical functor is the EZ spherical functor. The category on which the decomposition is constructed is a quotient of the category of a root stack. For this reason we shall also discuss, following [HLS16], when a periodic decomposition on a triangulated category descends to a periodic decomposition on a quotient. We conclude these notes with a few propositions of research directions related to spherical functors.

Acknowledgements I thank my youngest son, Tymek, for sleeping peacefully every morning from 8:30 till 9:30 AM, sometimes as long as 10 AM, and thus giving me time to work on these notes. I am also grateful to his siblings, Tosia and Staś, for effectively fighting all infections and keeping an (almost) perfect kindergarten attendance in the three months I took to complete these notes.

2. OVERVIEW

2.1. Spherical objects. The notion of a spherical object first appeared in the work of Seidel and Thomas [ST01]. It was defined in a more general set-up, but main examples of spherical objects are object of the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on a smooth projective variety.

Definition 2.1. Let X be a smooth projective variety of dimension n over a field k . An object $E \in \mathcal{D}^b(X)$ is *n-spherical* if

- (a) $\mathrm{Hom}_X^*(E, E) \simeq H^*(\mathbb{S}^n)$, i.e. $\mathrm{Hom}_X(E, E[i]) = \begin{cases} k & \text{if } i = 0, n \\ 0 & \text{else} \end{cases}$
- (b) There exists an isomorphism $E \otimes \omega_X \simeq E$.

Let X be a variety as above and let F in $\mathcal{D}^b(X)$ be any object. One can assign to it the twist functor $T_F: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X)$ whose value for any $G \in \mathcal{D}^b(X)$ fits into a distinguished triangle

$$F \otimes \mathrm{Hom}_X^*(F, G) \xrightarrow{\mathrm{ev}} G \rightarrow T_F(G),$$

where ev is the evaluation morphism.

The main result of [ST01] is that if E is spherical, then the twist functor T_E is an equivalence, see [ST01, Theorem 1.2].

The main examples of spherical functors are

Example 2.2. Let X be a strict Calabi-Yau variety. Then any locally free sheaf on X is spherical. The twist functor is the reflection functor. It is a Fourier-Mukai transform whose kernel is the shift of the ideal sheaf of the diagonal on $X \times X$.

Example 2.3. Let X be a surface and $C \subset X$ a smooth rational curve with $C \cdot C = -2$. Then $\mathcal{O}_C$ is spherical.

Example 2.4. Recently, Donovan and Zheng [DZ26] studied derived category of a weighted Hilbert scheme of $\mathbb{C}^3$ and the group of derived autoequivalences generated by twists in three spherical objects, the structure sheaves of exceptional surfaces.

Example 2.5. Let $f: X \rightarrow Y$ be a small contraction of a rational $(-1, -1)$ curve C in a threefold X . Then $\mathcal{O}_C$ is a spherical object.

Example 2.6. Let $X \subset \mathbb{P}^N$ be a smooth quasi-projective variety and let $i: Y \rightarrow X$ be an embedding of a complete connected hypersurface such that $i^*\omega_X$ is trivial. If $F \in \mathcal{D}^b(Y)$ is an exceptional object with a finite locally free resolution then i_*F is spherical.

Remark 2.7. The motivation for the name “spherical object” comes from the homological mirror symmetry. The mirror of a Calabi-Yau variety X should be a compact symplectic manifold (M, β) such that $\mathcal{D}^b(X) \simeq \mathcal{D}^b\text{Fuk}(M, \beta)$, where $\text{Fuk}(M, \beta)$ is the Fukaya category generated by Lagrangian submanifolds. It was conjectured in [ST01] that the spherical twist along an n -spherical object should correspond to a Dehn twist along a submanifold which is diffeomorphic to an n -sphere $\mathbb{S}^n$.

2.2. Spherical functors. One can view a spherical object $E \in \mathcal{D}^b(X)$ as a functor

$$F(-) = (-) \otimes_k E: \mathcal{D}^b(k) \rightarrow \mathcal{D}^b(X).$$

Its right adjoint is the functor

$$R(-) = \text{Hom}^*(E, -): \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(k).$$

As both X and k admit Serre functors, F has also the left adjoint

$$L(-) = \text{Hom}^*(E \otimes \omega_X^{-1}[-n], -): \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(k).$$

The evaluation map $E \otimes \text{Hom}^*(E, G) \rightarrow G$ is the $F \dashv R$ adjunction counit for $G \in \mathcal{D}^b(X)$. If one could consider cones of the natural transformations then the spherical twist would be the cone the adjunction counit $\varepsilon^R: FR \rightarrow \text{Id}_{\mathcal{D}^b(X)}$. Let us also consider the fiber C of the adjunction unit $\text{Id}_{\mathcal{D}^b(k)} \rightarrow RF$.

Lemma 2.8. *Object E is spherical if and only if $C = [-n-1]$ and there exists an isomorphism $R \simeq CL[1]$.*

Proof. Let us sketch the argument here. First, note that, since any object of $\mathcal{D}^b(k)$ is a direct sum of copies of shifts of k , any exact functor $\mathcal{D}^b(k) \rightarrow \mathcal{D}^b(k)$ is determined by its value on k . Consider the distinguished triangle

$$k \rightarrow \text{Hom}^*(E, E) \rightarrow C(k)[1]$$

given by the definition of C . As the first map maps k to the identity morphism of E , we see that $\text{Hom}^*(E, E) \simeq H^*(\mathbb{S}^n)$ if and only if $C \simeq [-n-1]$. By Yoneda, the condition $E \simeq E \otimes \omega_X$ is clearly equivalent to the fact that the functor

$$CL[-1] = \text{Hom}^*(E \otimes \omega_X^{-1}[-n], -)[-n] = \text{Hom}^*(E \otimes \omega_X^{-1}, -)$$

is isomorphic to R . □

The observation in the above Lemma led R. Anno to define spherical functors in [Ann13]. With time more definitions of spherical functors appeared.

The general set-up needed to define spherical functors between triangulated categories is clear. Let $\mathcal{A}, \mathcal{B}$ be triangulated categories and $F: \mathcal{A} \rightarrow \mathcal{B}$ an exact functor with left and right adjoints $L \dashv F \dashv R$. We assume that

(AS1) F , its adjoint functors, the identity functors on $\mathcal{A}, \mathcal{B}$, their compositions and adjunction units and counits are objects and morphisms of a triangulated category.

(AS2) The composition of a distinguished triangle of functors with a functor yields a distinguished triangle.

Assumption (AS1) allows one to consider cones of the adjunction unit and counit. In particular, one can define the following endofunctors via distinguished triangles

$$\begin{aligned}
FR &\xrightarrow{\varepsilon^R} \text{Id}_{\mathcal{B}} \rightarrow T \xrightarrow{\alpha} FR[1] && \text{the twist } T, \\
C &\rightarrow \text{Id}_{\mathcal{A}} \xrightarrow{\eta^R} RF \xrightarrow{\beta} C[1] && \text{the cotwist } C, \\
T' &\rightarrow \text{Id}_{\mathcal{B}} \xrightarrow{\eta^L} FL \xrightarrow{\gamma} T'[1] && \text{the dual twist } T', \\
LF &\xrightarrow{\varepsilon^L} \text{Id}_{\mathcal{A}} \rightarrow C' \xrightarrow{\delta} LF[1] && \text{the dual cotwist } C',
\end{aligned}$$

where ε^L , η^L , respectively ε^R , η^R , are the counit and unit of the $L \dashv F$, respectively $F \dashv R$ adjunction.

Before stating various definitions of spherical functors, we note the following relations between natural transformations. The proof will be discussed in the next Section.

Lemma 2.9. [AL17, Kuz19] *The following conditions are equivalent:*

(a) *Natural transformation*

$$(1) \quad L \oplus R \xrightarrow{(\eta_L^R, R\eta^L)} RFL$$

is an isomorphism.

(b) *Natural transformation*

$$(2) \quad R \xrightarrow{R\eta^L} RFL \xrightarrow{\beta_L} CL[1]$$

is an isomorphism.

(c) *Natural transformation*

$$(3) \quad L \xrightarrow{\eta_L^R} RFL \xrightarrow{R\gamma} RT'[1]$$

is an isomorphism.

Lemma 2.10. *The following conditions are equivalent:*

(a) *Natural transformation*

$$(4) \quad LFR \xrightarrow{\begin{pmatrix} L\varepsilon^R \\ \varepsilon_R^L \end{pmatrix}} L \oplus R$$

is an isomorphism.

(b) *Natural transformation*

$$(5) \quad LT[-1] \xrightarrow{L\alpha[-1]} LFR \xrightarrow{\varepsilon_R^L} R$$

is an isomorphism.

(c) *Natural transformation*

$$(6) \quad C'R[-1] \xrightarrow{\delta_R[-1]} LFR \xrightarrow{L\varepsilon^R} R$$

is an isomorphism.

Note that conditions (1) and (4) are, in practice, easier to verify than the remaining ones as they do not include the connecting morphisms in the distinguished triangles.

Now, we are ready to recall various definitions of spherical functors

We say that functor F is *spherical* if

- [Ann13, Mea21, Plo05]: C is an equivalence and (2) is an isomorphism.
- [Kuz19]: Natural transformations (1) and (4) are isomorphisms.

- [Add16]: C is an equivalence and there exists an isomorphism $R \simeq CL[1]$.

The main statement is about spherical functors read: If F is spherical then

- [Ann13, Mea21, Plo05]: T is an equivalence with inverse T' .
- [Kuz19]: T, T', C, C' are equivalences, pair-wise quasi-inverse.
- [Add16]: T is an equivalence.

In fact, to conclude that T is an equivalence from the condition on C and CL one also needs to assume that

(AS3) there are pairs of adjoint functors $T' \dashv T, C' \dashv C$

Assumptions (AS1)-(AS3) are satisfied if one considers Fourier-Mukai transforms between smooth projective varieties.

Examples of spherical functors shall be discussed in detail later in the text. Let us just mention a couple here. In the examples all schemes are assumed to be smooth projective.

Example 2.11. Let $i: D \rightarrow X$ be an inclusion of an effective Cartier divisor. Then functor $i_*: \mathcal{D}^b(D) \rightarrow \mathcal{D}^b(X)$ is spherical.

Example 2.12. Suppose that $D \subset X$ is an effective Cartier divisor such that $D \simeq \mathbb{P}(E)$ has a structure of projective vector bundle of rank r over some variety Z , $\pi: \mathbb{P}(E) \rightarrow Z$. Then $i_*\pi^*: \mathcal{D}^b(Z) \rightarrow \mathcal{D}^b(X)$ is spherical if $\mathcal{O}_D(D) \simeq \pi^*L \otimes \mathcal{O}_\pi(-r)$, for some invertible sheaf L on Z .

Example 2.13. Let $f: X \rightarrow Y$ be a double cover. Then $Rf_*: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ is spherical.

2.3. Spherical DG functors. Anno and Logvinenko in [AL17] proposed to define spherical functors as DG functors between DG categories to avoid the problem of non-functorial cone.

They consider small DG categories $\mathcal{A}, \mathcal{B}$. Then a quasi-functor $F: \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{B})$ is given by an $\mathcal{A} - \mathcal{B}$ bimodule M which is perfect as $\mathcal{B}$ module. If we assume further that M is also perfect as an $\mathcal{A}$ module, then its derived duals $M^{\mathcal{B}}, M^{\mathcal{A}}$ give functors left and right adjoint to F . The authors construct explicit trace and action morphisms of DG bimodules. They lift the adjunction unit and counit for the induced functors of homotopy categories. The twist, dual twist, cotwist and dual cotwist are then given by bimodules obtained as (functorial) cones of the trace and action maps.

Definition 2.14. An $\mathcal{A} - \mathcal{B}$ bimodule M is *spherical* if it is $\mathcal{A}$ and $\mathcal{B}$ perfect and

- T and T' are quasi-inverse equivalences of $\mathcal{D}(\mathcal{B})$.
- C and C' are quasi-inverse equivalences of $\mathcal{D}(\mathcal{A})$.
- The natural transformation (2) is an isomorphism.
- The natural transformation (5) is an isomorphism.

Theorem 2.15. [AL17, Theorem 5.1] *Let M be an $\mathcal{A}$ and $\mathcal{B}$ perfect $\mathcal{A} - \mathcal{B}$ bimodule. If any two of the above conditions hold then all four hold and $F(-) = (-) \otimes M$ is spherical.*

In practice, checking that a functor F is spherical in the above sense requires considering an explicit lift to a quasi-functor of DG enhancements. For example, assume that X and Y are separated schemes of finite type over a field k . One can enhance $\mathcal{D}^b(\text{Coh}(X))$ and $\mathcal{D}^b(\text{Coh}(Y))$ by smooth DG algebras $\mathcal{A}$ and $\mathcal{B}$. Then the bounded derived category of coherent sheaves on $X \times Y$ is enhanced by the DG algebra $\mathcal{A}^{\text{pp}} \otimes \mathcal{B}$ and the Fourier-Mukai functors $\mathcal{D}^b(\text{Coh}(X)) \rightarrow \mathcal{D}^b(\text{Coh}(Y))$ correspond to $\mathcal{A} - \mathcal{B}$ bimodules that are $\mathcal{B}$ perfect. With [AL17, Conjecture 5.15] Anno and Logvinenko *conjecture* that the trace and action maps on DG level descend to the explicit morphisms between Fourier-Mukai kernels of the identity morphism and composition of F with its adjoints.

2.4. Semi-orthogonal decompositions and mutations. Before discussing the definition of triangle-spherical functors and the theory that motivated it, we recall the notion of a semi-orthogonal decomposition and mutation functors.

Let $\mathcal{T}$ be a triangulated category. Recall, [Bon89] that a semi-orthogonal decomposition (SOD) for $\mathcal{T}$ is the data of

- (a) Two full triangulated subcategories $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ such that $\text{Hom}_{\mathcal{T}}(\mathcal{B}, \mathcal{A}) = 0$ and any $T \in \mathcal{T}$ fits into a distinguished triangle $B_T \rightarrow T \rightarrow A_T \rightarrow B_T[1]$ with $A_T \in \mathcal{A}$ and $B_T \in \mathcal{B}$.
- (b) A fully faithful functor $i_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{T}$ of triangulated categories with left adjoint $i_{\mathcal{A}}^*$.
- (c) A fully faithful functor $i_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{T}$ of triangulated categories with right adjoint $i_{\mathcal{B}}^!$.

We denote such SOD as

$$(7) \quad \mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle.$$

Lemma 2.16. *The data (a)-(c) above are equivalent.*

Proof. Assume that we have an SOD in the sense (a). One checks that objects A_T, B_T are functorial in T , hence $T \mapsto A_T$ is a functor $i_{\mathcal{A}}^*: \mathcal{T} \rightarrow \mathcal{A}$ and $T \mapsto B_T$ is a functor $i_{\mathcal{B}}^!: \mathcal{T} \rightarrow \mathcal{B}$. Further, a direct verification shows that the functors are adjoint to the embeddings of full subcategories.

Assume now that we have an SOD in the sense of (b). Then we define

$$\mathcal{B} = {}^{\perp}\mathcal{A} = \{T \in \mathcal{T} \mid \text{Hom}(T, \mathcal{A}) = 0\}.$$

For any $T \in \mathcal{T}$ the adjunction unit $T \rightarrow i_{\mathcal{A}}i_{\mathcal{A}}^*T$ fits into a distinguished triangle

$$B_T \rightarrow T \rightarrow i_{\mathcal{A}}i_{\mathcal{A}}^*T$$

with $B_T \in \mathcal{B}$, hence it gives an SOD in the sense of (a).

Given an SOD in the sense of (c), the category $\mathcal{A}$ is defined analogously as

$$\mathcal{A} = \mathcal{B}^{\perp} = \{T \in \mathcal{T} \mid \text{Hom}(\mathcal{B}, T) = 0\}.$$

□

The proof of the above Lemma tells us that given an SOD (7) we have a functorial distinguished triangle

$$(8) \quad i_{\mathcal{B}}i_{\mathcal{B}}^! \xrightarrow{\varepsilon_{\mathcal{B}}} \text{Id} \xrightarrow{\eta^{\mathcal{A}}} i_{\mathcal{A}}i_{\mathcal{A}}^* \rightarrow i_{\mathcal{B}}i_{\mathcal{B}}^*[1].$$

Note that the semi-orthogonality of $\mathcal{A}$ and $\mathcal{B}$ can be expressed as

$$(9) \quad i_{\mathcal{A}}^*i_{\mathcal{B}} = 0, \quad i_{\mathcal{B}}^!i_{\mathcal{A}} = 0.$$

Remark 2.17. One is tempted think of an object $T \in \mathcal{T}$ as the cone of an element in the gluing bifunctor $\text{Hom}(i_{\mathcal{A}}(-), i_{\mathcal{B}}[1](-)): \mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \text{Vect}$.

We say that the subcategory $\mathcal{A} \subset \mathcal{T}$ is left admissible (the inclusion admits left adjoint), while $\mathcal{B} \subset \mathcal{T}$ is right admissible.

Assume now that $\mathcal{B} \subset \mathcal{T}$ is admissible, i.e. $i_{\mathcal{B}}$ admits both right $i_{\mathcal{B}}^!$ and left $i_{\mathcal{B}}^*$ adjoint functors. Then category $\mathcal{T}$ admits an extra SOD:

$$\mathcal{T} = \langle \mathcal{B}, \mathcal{C} \rangle.$$

Lemma 2.18. [Bon89] *The right mutation functor $R_{\mathcal{B}} = i_{\mathcal{C}}^!i_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{C}$ is an equivalence whose quasi-inverse is the left mutation functor $L_{\mathcal{B}} = i_{\mathcal{A}}^*i_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{A}$.*

Proof. One checks that $R_{\mathcal{B}}L_{\mathcal{B}} \simeq \text{Id}_{\mathcal{C}}$ and $L_{\mathcal{B}}R_{\mathcal{B}} \simeq \text{Id}_{\mathcal{A}}$. We check the first isomorphism, the proof of the second one is analogous.

Composing the functorial distinguished triangle (8) with $i_{\mathcal{C}}$ on the right and $i_{\mathcal{C}}^!$ on the left gives a distinguished triangle

$$i_{\mathcal{C}}^!i_{\mathcal{B}}i_{\mathcal{B}}^!i_{\mathcal{C}} \rightarrow i_{\mathcal{C}}^!i_{\mathcal{C}} \rightarrow i_{\mathcal{C}}^!i_{\mathcal{A}}i_{\mathcal{A}}^*i_{\mathcal{C}} \rightarrow i_{\mathcal{C}}^!i_{\mathcal{B}}i_{\mathcal{B}}^*[1].$$

As $i_{\mathcal{C}}$ is fully faithful, $i_{\mathcal{C}}^! i_{\mathcal{C}} \simeq \text{Id}_{\mathcal{C}}$. Moreover, the semi-orthogonality of $\langle \mathcal{B}, \mathcal{C} \rangle$ implies that $i_{\mathcal{C}}^! i_{\mathcal{B}} = 0$. Hence, the above triangle is isomorphic to the distinguished triangle

$$0 \longrightarrow \text{Id}_{\mathcal{C}} \longrightarrow R_{\mathcal{B}} L_{\mathcal{B}} \longrightarrow 0.$$

□

Note that, an admissible subcategory yields extra adjoint functors.

Lemma 2.19. *Given an admissible subcategory $\mathcal{B} \subset \mathcal{T}$ functor $i_{\mathcal{C}} i_{\mathcal{A}}^! : \mathcal{A} \longrightarrow \mathcal{T}$ is left adjoint to $i_{\mathcal{A}}^*$.*

Proof. Consider $A \in \mathcal{A}$ and $T \in \mathcal{T}$. We have

$$\begin{aligned} \text{Hom}_{\mathcal{T}}(i_{\mathcal{C}} i_{\mathcal{A}}^!(A), T) &\simeq \text{Hom}_{\mathcal{C}}(i_{\mathcal{C}}^! i_{\mathcal{A}}(A), i_{\mathcal{C}}^!(T)) \simeq \text{Hom}_{\mathcal{A}}(i_{\mathcal{A}}^* i_{\mathcal{C}} i_{\mathcal{A}}^!(A), i_{\mathcal{A}}^* i_{\mathcal{C}}^!(T)) \\ &\simeq \text{Hom}_{\mathcal{A}}(A, i_{\mathcal{A}}^* i_{\mathcal{C}} i_{\mathcal{A}}^!(T)) \simeq \text{Hom}_{\mathcal{A}}(A, i_{\mathcal{A}}^*(T)), \end{aligned}$$

where the first isomorphism follows from the $i_{\mathcal{C}} \dashv i_{\mathcal{C}}^!$ adjunction, the second isomorphism follows from the fact that $L_{\mathcal{B}} = i_{\mathcal{A}}^* i_{\mathcal{C}}$ is an equivalence. As $R_{\mathcal{B}} = i_{\mathcal{C}}^! i_{\mathcal{A}}$ is the quasi-inverse of $L_{\mathcal{B}}$, the third isomorphism follows, see Lemma 2.18. Finally, vanishing of $i_{\mathcal{A}}^* i_{\mathcal{B}}$, (9), implies that the $i_{\mathcal{C}} \dashv i_{\mathcal{C}}^!$ adjunction unit composed with $i_{\mathcal{A}}^*$ is an isomorphism. Indeed, its cone is $i_{\mathcal{A}}^* i_{\mathcal{B}} i_{\mathcal{B}}^* \simeq 0$. The last isomorphism follows. □

Analogously one proves

Lemma 2.20. *Given an admissible subcategory $\mathcal{B} \subset \mathcal{T}$ functor $i_{\mathcal{A}} i_{\mathcal{A}}^* i_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{T}$ is right adjoint to $i_{\mathcal{C}}^!$.*

Remark 2.21. Note that in the presence of the extra adjoint to $i_{\mathcal{B}}$, the gluing bifunctor is given by the functor $i_{\mathcal{B}}^* i_{\mathcal{A}}[-1] : \mathcal{A} \longrightarrow \mathcal{B}$, which we shall refer to as the *gluing* functor:

$$\text{Hom}_{\mathcal{T}}(i_{\mathcal{A}}(A), i_{\mathcal{B}}(B)[1]) = \text{Hom}_{\mathcal{B}}(i_{\mathcal{B}}^* i_{\mathcal{A}}(A)[-1], B)$$

If $\mathcal{A}$ is admissible, i.e if $i_{\mathcal{A}}^!$ exists, the gluing bifunctor is given by $i_{\mathcal{A}}^! i_{\mathcal{B}}[1] : \mathcal{B} \longrightarrow \mathcal{A}$ which we shall refer to as the *co-gluing* functor. Note that the co-gluing functor is right adjoint to the gluing one.

2.5. Gluing via spherical functors. In [HLS16] Halpern-Leistner and Shipman considered a balanced wall crossing in the theory of variations of GIT quotients for a reductive group G acting on a smooth, projective over affine scheme X . They noted that an explicit subcategory $\mathcal{C}_{\omega}$ of $\mathcal{D}^b(\text{Coh}(X/G))$ admits a four-periodic decomposition. This observation led to the following general theory.

Consider a pre-triangulated DG category $\mathcal{T}$. Assume that it admits a four-periodic semi-orthogonal decomposition, i.e. $\mathcal{A}, \dots, \mathcal{D}$ are subcategories of $\mathcal{T}$ that give the followings SOD's of the homotopy category of $\mathcal{T}$:

$$\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{B}, \mathcal{C} \rangle = \langle \mathcal{C}, \mathcal{D} \rangle = \langle \mathcal{D}, \mathcal{A} \rangle.$$

Then the gluing functor $i_{\mathcal{B}}^* i_{\mathcal{A}}$ is spherical, see [HLS16, Theorem 3.11].

In the opposite direction, let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a spherical DG functor. Then the category $\mathcal{T} = \mathcal{A} \times_F \mathcal{B}$ glued from $\mathcal{A}$ and $\mathcal{B}$ via F admits a 4-periodic semiorthogonal decomposition as above.

We shall discuss details of this construction in Section 3.4.

2.6. Triangle spherical functors. Kapranov and Schechtman in [KS14] first observed that a four-periodic decomposition (or, more generally, a spherical pair) in a triangulated category allows to consider spherical functors without any additional assumptions on the category of exact functors between $\mathcal{A}$ and $\mathcal{B}$.

The first example of such a decomposition was given by Bodzenta and Bondal in [BB22] which yield to an example of a spherical functor whose source category was the bounded derived

category of finite dimensional modules over a noncommutative Artinian algebra. This spherical functor is a vast generalisation of the spherical object in Example 2.3.

In [BD24] Bodzenta and Donovan introduced the notion of a *triangle spherical* functor. It is defined as the gluing functor $i_{\mathcal{B}}^* i_{\mathcal{A}*}$ for a four-periodic semi-orthogonal decomposition

$$(10) \quad \mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{B}, \mathcal{C} \rangle = \langle \mathcal{C}, \mathcal{D} \rangle = \langle \mathcal{D}, \mathcal{A} \rangle.$$

They also proved that the root stack $\sqrt{X/D}$ for an effective Cartier divisor D in a reduced noetherian separated k scheme X over a field k of characteristic zero admits a 4-periodic SOD with the gluing functor $i_{D*}: \mathcal{D}^b(D) \rightarrow \mathcal{D}^b(X)$.

Remark 2.22. More generally, Dyckerhoff, Kapranov and Schechtman in [DKS23] introduced N spherical functors to be gluing functors for N -periodic SODs of stable infinity categories.

3. DEFINITIONS OF SPHERICAL FUNCTORS

3.1. Spherical functors between triangulated categories. We recall Kuznetsov's proof [Kuz19] that given a spherical functor in the sense of Kuznetsov, functors T and C are equivalences. We also sketch the arguments given by Addington, Meachan and Ploog [Add16, Mea21, Plo05].

Recall that we work under two assumptions (AS1) and (AS2).

First, we prove Lemmas 2.9 and 2.10.

Proof of Lemma 2.9. Consider the following morphism of distinguished triangles. The top row is distinguished by assumption (AS2), the lower one is the distinguished triangle of the direct sum and the vertical arrows are the natural transformations in the formulation of the Lemma.

$$\begin{array}{ccccc} L & \xrightarrow{\eta_L^R} & RFL & \xrightarrow{\beta_L} & CL[1] \\ \text{Id} \uparrow & & (1) \uparrow & & (2) \uparrow \\ L & \longrightarrow & L \oplus R & \longrightarrow & R \end{array}$$

The two-out-of-three property implies that condition (a) is equivalent to (b).

The equivalence of (a) and (c) follows from the following morphism of distinguished triangles.

$$\begin{array}{ccccc} R & \xrightarrow{R\eta^L} & RFL & \xrightarrow{R\gamma} & RT'[1] \\ \text{Id} \uparrow & & (1) \uparrow & & (3) \uparrow \\ R & \longrightarrow & L \oplus R & \longrightarrow & L \end{array}$$

□

Proof of Lemma 2.10. Analogously as in the proof of the previous lemma equivalence of (a) and (b) follows from the following morphism of distinguished triangles.

$$\begin{array}{ccccc} R & \longrightarrow & L \oplus R & \longrightarrow & R \\ (5) \uparrow & & (4) \uparrow & & \text{Id} \uparrow \\ LT[-1] & \xrightarrow{L\alpha[-1]} & LFR & \xrightarrow{L\epsilon^R} & L \end{array}$$

The equivalence of (a) and (c) follows from the following morphism:

$$\begin{array}{ccccc} L & \longrightarrow & L \oplus R & \longrightarrow & R \\ (6) \uparrow & & (4) \uparrow & & \text{Id} \uparrow \\ C'R[-1] & \xrightarrow{\delta_R[-1]} & LFR & \xrightarrow{\epsilon_R^L} & R \end{array}$$

□

Proposition 3.1. *If equivalent conditions of Lemma 2.9 hold then $TT' \simeq \text{Id}_{\mathcal{B}}$ and $CC' \simeq \text{Id}_{\mathcal{A}}$.*

Proof. Isomorphism (3) composed with F contributes the top row of the following commutative diagram

$$\begin{array}{ccccc}
 FL & \xrightarrow{F\eta_L^R} & FRFL & \xrightarrow{FR\gamma} & FRT'[1] \\
 & \searrow \text{Id} & \downarrow \varepsilon_{FL}^R & & \downarrow \varepsilon_{T'[1]}^R \\
 & & FL & \xrightarrow{\gamma} & T'[1]
 \end{array}$$

The external commutative square gives a morphism of distinguished triangles

$$\begin{array}{ccccc}
 T[-1]T'[1] & \longrightarrow & FRT'[1] & \xrightarrow{\varepsilon_{T'[1]}^R} & T'[1] \\
 \uparrow & & \uparrow F(3) & & \uparrow \text{Id} \\
 \text{Id}_{\mathcal{B}} & \longrightarrow & FL & \xrightarrow{\gamma} & T'[1]
 \end{array}$$

The two-out-of-three property implies that $TT' \simeq \text{Id}_{\mathcal{B}}$.

Isomorphism (2) composed with F gives the top row of the following commutative diagram

$$\begin{array}{ccccc}
 RF & \xrightarrow{R\eta_F^L} & RFLF & \xrightarrow{\beta_{LF}} & CLF[1] \\
 & \searrow \text{Id} & \downarrow RF\varepsilon^L & & \downarrow C\varepsilon^L[1] \\
 & & RF & \xrightarrow{\beta} & C[1]
 \end{array}$$

The external commutative square fits into a morphism of distinguished triangles

$$\begin{array}{ccccc}
 C[1]C'[1] & \longrightarrow & CLF[1] & \xrightarrow{C\varepsilon^L[1]} & C[1] \\
 \uparrow & & \uparrow (2)F & & \uparrow \text{Id} \\
 \text{Id} & \longrightarrow & RF & \xrightarrow{\beta} & C[1]
 \end{array}$$

The two-out-of-three property implies that $CC' \simeq \text{Id}$. □

Proposition 3.2. *If equivalent conditions of Lemma 2.10 hold then $C'C \simeq \text{Id}_{\mathcal{A}}$ and $T'T \simeq \text{Id}_{\mathcal{B}}$.*

Proof. Isomorphism (5) composed with F fits into a commutative diagram

$$\begin{array}{ccccc}
 FLT[-1] & \xrightarrow{FL\alpha[-1]} & FLFR & \xrightarrow{F\varepsilon_R^L} & FR \\
 \eta_{T[-1]}^L \uparrow & & \uparrow \eta_{FR}^L & \nearrow \text{Id} & \\
 T[-1] & \xrightarrow{\alpha[-1]} & FR & &
 \end{array}$$

The external commutative square yields a morphism of distinguished triangles

$$\begin{array}{ccccc}
 T[-1] & \xrightarrow{\alpha[-1]} & FR & \longrightarrow & \text{Id} \\
 \uparrow \text{Id} & & \uparrow F(5) & & \uparrow \\
 T[-1] & \xrightarrow{\eta_{T[-1]}^L} & FLT[-1] & \xrightarrow{\gamma_{T[-1]}} & T'[1]T[-1]
 \end{array}$$

The two-out-of-three property implies that $T'T \simeq \text{Id}$.

Isomorphism (6) composed with F gives the top row of the following commutative diagram

$$\begin{array}{ccccc}
 C'RF[-1] & \xrightarrow{\delta_{RF}[-1]} & LFRF & \xrightarrow{L\varepsilon_F^R} & LF \\
 C'\eta^R[-1] \uparrow & & \uparrow LF\eta^R & \nearrow \text{Id} & \\
 C'[-1] & \longrightarrow & LF & &
 \end{array}$$

The external commutative square yields a morphism of distinguished triangles

$$\begin{array}{ccccc} C'[-1] & \longrightarrow & LF & \xrightarrow{\varepsilon^L} & \text{Id} \\ \uparrow \text{Id} & & \uparrow (6)^F & & \uparrow \\ C'[-1] & \xrightarrow{C'\eta^R[-1]} & C'RF[-1] & \longrightarrow & C'[-1]C[1] \end{array}$$

The two-out-of-three property implies that $C'C \simeq \text{Id}$. $\square$

Theorem 3.3. *Assume (AS1) and (AS2) hold. If functor F is spherical in the sense of Kuznetsov then T and C are equivalences with quasi-inverses T' and C' .*

Proof. This follows immediately from Propositions 3.1 and 3.2. $\square$

Let now $F: \mathcal{A} \rightarrow \mathcal{B}$ be a spherical functor in the sense of Anno, Meachan and Ploog, i.e. assume that (2) is an isomorphism and C is an auto-equivalence of $\mathcal{A}$.

Note that we have

Lemma 3.4. *For any $B, B' \in \mathcal{B}$, there is an isomorphism*

$$\text{Hom}_{\mathcal{B}}(T'(B'), B) \simeq \text{Hom}_{\mathcal{B}}(B', T(B)).$$

Proof. For any $B, B' \in \mathcal{B}$, $\text{Hom}_{\mathcal{B}}(T'(B'), B)$ is the cone of the morphism $\text{Hom}_{\mathcal{B}}(FL(B'), B[1]) \rightarrow \text{Hom}_{\mathcal{B}}(B', B[1])$ given by the composition with $\eta_{B'}^L$. On the other hand $\text{Hom}_{\mathcal{B}}(B', T(B))$ is the cone of the morphism $\text{Hom}_{\mathcal{B}}(B'[-1], FR(B)) \rightarrow \text{Hom}_{\mathcal{B}}(B'[-1], B)$ given by the composition with ε_B^R . Hence, the sought isomorphism follows from the commutativity of the diagram

$$(11) \quad \begin{array}{ccc} \text{Hom}_{\mathcal{B}}(FL(B'), B) & \xrightarrow{(-) \circ \eta_{B'}^L} & \text{Hom}_{\mathcal{B}}(B', B) \\ \uparrow & \nearrow \varepsilon_B^R \circ (-) & \\ \text{Hom}_{\mathcal{B}}(B', FR(B)) & & \end{array}$$

where the vertical arrow maps $B' \xrightarrow{f} FR(B)$ to

$$FL(B') \xrightarrow{FL(f)} FLFR(B) \xrightarrow{F(\varepsilon_{R(B)}^L)} FR(B) \xrightarrow{\varepsilon_B^R} B.$$

The commutativity of (11) follows from the commutativity of the diagram:

$$\begin{array}{ccccc} FL(B') & \xrightarrow{FL(f)} & FLFR(B) & \xrightarrow{F(\varepsilon_{R(B)}^L)} & FR(B) \\ \eta_{B'}^L \uparrow & & \eta_{FR(B)}^L \uparrow & \nearrow \text{Id} & \downarrow \varepsilon_B^R \\ B' & \xrightarrow{f} & FR(B) & \xrightarrow{\varepsilon_B^R} & B. \end{array}$$

$\square$

Note that, in general, the isomorphisms in Lemma 3.4 do not extend to a pair of adjoint functors as the axioms of a triangulated category do not guarantee that they can be made functorial in B and B' .

To prove that a functor F spherical in the sense of Anno, Meachan and Ploog yields an equivalence T , we need to further assume that

(AS3) there are pairs of adjoint functors $T' \dashv T$, $C' \dashv C$

Meachan [Mea21] proposes a different “triangulated” proof the isomorphism $TT' \simeq \text{Id}_{\mathcal{B}}$, i.e. an argument analogous to the one given in [Kuz19] and recalled above. Addington, following [Plo05], checks directly on a spanning class $\text{Im}(F) \cup \ker(R)$ that T is fully faithful, hence $T'T \simeq \text{Id}_{\mathcal{B}}$. Note that we need the isomorphism $R \simeq CL[1]$ to prove that it is a spanning class. Then the fact that T is an equivalence follows from the lemma

Lemma 3.5. [Huy06, Lemma 1.50, Exercise 1.51] *Let $S: \mathcal{T} \rightarrow \mathcal{T}'$ be a fully faithful exact functor between triangulated categories.*

- (a) *Assume that S admits right adjoint $S \dashv R$. Then S is an equivalence if and only if for any $T' \in \mathcal{T}'$ isomorphism $R(T') \simeq 0$ implies that $T' \simeq 0$.*
- (b) *Assume that S admits left adjoint $L \dashv S$. Then S is an equivalence if and only if for any $T' \in \mathcal{T}'$ isomorphism $L(T') \simeq 0$ implies that $T' \simeq 0$.*

Remark 3.6. Meachan in [Mea21, Proposition 3.2] proves that if C is an equivalence and there exists an isomorphism $R \simeq CL[1]$ then (2) is an isomorphism.

3.2. Examples when the assumptions are satisfied. Now we give examples under which one can consider spherical functors between triangulated categories, i.e. we discuss when conditions (AS1)-(AS3) hold.

We begin by defining the spherical twist for a spherical object. The greatest generality considered by Seidel and Thomas in [ST01] is that of an abelian category $\mathcal{S}$ and a full subcategory $\mathcal{S}'$ such that

- (a) $\mathcal{S}' \subset \mathcal{S}$ is a Serre subcategory,
- (b) $\mathcal{S}$ has arbitrary direct sums and products,
- (c) $\mathcal{S}$ has enough injectives and direct sum of injective objects is again injective,
- (d) For any epimorphism $f: A \rightarrow A'$ with $A \in \mathcal{S}$ and $A' \in \mathcal{S}'$ there exists $B' \in \mathcal{S}'$ and $g: B' \rightarrow A$ such that $fg: B' \rightarrow A$ is still an epimorphism.

Note that if X is a noetherian scheme over a field k then categories $\text{Coh}(X) \subset \text{QCoh}(X)$ satisfy the above conditions.

The authors further consider category $\mathfrak{k}$ of bounded below cochain complexes C of $\mathcal{S}$ -injective objects such that $H^i(C) \in \mathcal{S}'$ for all i and $H^i(C) = 0$ for $i \gg 0$. Note that $\mathfrak{k}$ is equivalent to $\mathcal{D}^b(\mathcal{S}')$.

In this set-up, an object $E \in \mathfrak{k}$ is n -spherical, if

- (a) E is a bounded complex.
- (b) For any $F \in \mathfrak{k}$ both $\text{Hom}^*(E, F)$ and $\text{Hom}^*(F, E)$ have finite global dimension over k .
- (c) $\text{Hom}^i(E, E) = \begin{cases} k & \text{if } i = 0, n \\ 0 & \text{else.} \end{cases}$
- (d) The composition $\text{Hom}^j(F, E) \times \text{Hom}^{n-j}(E, F) \rightarrow \text{Hom}^n(E, E) \simeq k$ is a non-degenerate pairing for any $F \in \mathfrak{k}$, any $j \in \mathbb{Z}$.

Given a spherical object $E \in \mathfrak{k}$, the twist functor is defined as the total complex

$$T_E(F) = \text{Tot}(\text{Hom}(E, F) \otimes_k E \rightarrow F)$$

of the evaluation morphism.

If X is smooth projective the picture greatly simplifies. Given a spherical object $E \in \mathcal{D}^b(X)$ the twist functor as defined above is isomorphic to the Fourier-Mukai transform given by

$$P = \text{Cone}(E^* \boxtimes E \xrightarrow{\eta} \mathcal{O}_\Delta)$$

on $X \times X$, where η corresponds under the $i^* \dashv i_*$ adjunction to the evaluation morphism $E^* \otimes E \simeq i^*(E^* \boxtimes E) \rightarrow \mathcal{O}_X$. Here, $i: X \rightarrow X \times X$ is the diagonal embedding and $\mathcal{O}_\Delta = i_* \mathcal{O}_X$.

To define spherical functors one can follow both the above approaches. First, one can consider Fourier-Mukai transforms between bounded derived categories of coherent sheaves on smooth projective varieties, see [Add16, Mea21, Kuz19]. Why so strong assumptions? Because we want the left and right adjoint to a Fourier-Mukai transform to be Fourier-Mukai transforms as well.

Rouquier in [Rou04] proposed a more general approach which yields functorial cones. It works for algebraic triangulated categories, i.e. stable categories $\bar{\mathcal{E}}$ of Frobenius exact categories $\mathcal{E}$, see Remarks 3.7 and 3.8 below.

If $\mathcal{E}$ is a Frobenius category, let $\text{Comp}_{\text{acyc}}(\mathcal{E}\text{-proj})$ be the Frobenius category of acyclic complexes of projective objects in $\mathcal{E}$. Note that if Frob is a 2-category of Frobenius categories,

exact functors preserving projectives and natural transformations, then $\mathcal{E} \mapsto \text{Comp}_{\text{acyc}}(\mathcal{E}\text{-proj})$ is an endofunctor of Frob . Moreover, if Tr is the 2-category of triangulated categories, exact functors and natural transformations, then the functor $\text{Frob} \rightarrow \text{Tr}$, $\mathcal{E} \mapsto \bar{\mathcal{E}}$ factors via $\mathcal{E} \mapsto \text{Comp}_{\text{acyc}}(\mathcal{E}\text{-proj})$.

The category of exact functors $\text{Comp}_{\text{acyc}}(\mathcal{E}\text{-proj}) \rightarrow \text{Comp}_{\text{acyc}}(\mathcal{E}'\text{-proj})$ which preserve projectives is again a Frobenius category. We denote by $\text{AlgTr}(\mathcal{E}, \mathcal{E}')$ its stable category. This yields a 2-category AlgTr of algebraic triangulated categories whose objects are Frobenius categories and morphisms are $\text{AlgTr}(\mathcal{E}, \mathcal{E}')$ as above.

There is a 2-functor $\text{AlgTr} \rightarrow \text{Tr}$ which maps $\mathcal{E}$ to $\bar{\mathcal{E}}$. It is 2-fully-faithful. This allows one to consider cones of natural transformations of functors. Indeed, a natural transformation of functors in $\text{AlgTr}(\mathcal{E}, \mathcal{E}')$ is a morphism in a triangulated category, hence it has a cone.

Remark 3.7. Recall that a Frobenius category $\mathcal{E}$ is an exact category (additive category with a class of ‘short exact sequences’) which has enough projective and injective objects and such that the class of projective objects in $\mathcal{E}$ coincides with the class of injectives. Stable category $\bar{\mathcal{E}}$ of $\mathcal{E}$ has the same objects as $\mathcal{E}$ while morphisms in $\bar{\mathcal{E}}$ are defined as the quotient of morphisms in $\mathcal{E}$ by the subgroups of morphisms which factor via projective objects. The stable category of a Frobenius category admits a canonical structure of a triangulated category [Kel96].

Remark 3.8. Recall that given an additive category $\mathcal{A}$, the category of complexes $C(\mathcal{A})$ admits a structure of a Frobenius category. Conflations are $C_1 \xrightarrow{i} C \xrightarrow{p} C_2$ such that (i^n, p^n) is a split exact sequence for any i , see [Kel96]. The projective-injective objects in $C(\mathcal{A})$ are then complexes which are homotopic to zero. The stable category is the homotopy category of complexes over $\mathcal{A}$.

If $\mathcal{A}$ is abelian, the derived category $\mathcal{D}(\mathcal{A})$ is then the stable category of the Frobenius category of complexes of projective or injective objects in $\mathcal{A}$ (which is an additive category).

3.3. DG gluing. The non-functoriality of the cone prevents us from forming a triangulated category from the data of triangulated categories $\mathcal{A}, \mathcal{B}$ and a gluing bifunctor $\mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \text{Vect}$. However, such a construction is possible in the world of DG categories. We recall the definitions and main properties following [KL15].

Consider DG categories $\mathcal{A}, \mathcal{B}$ over a field k and a bifunctor $\Phi: \mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \mathcal{V}\text{ect}$, where $\mathcal{V}\text{ect}$ is the DG category of complexes of k -vector spaces.

One defines the DG category $\mathcal{T} = \mathcal{A} \times_{\Phi} \mathcal{B}$. Its objects are $(A \in \mathcal{A}, B \in \mathcal{B}, \varphi \in \Phi(A, B)^0)$ where φ is a closed element of degree zero. Morphisms in $\mathcal{T}$ are defined as cones in the category of complexes, i.e. we have a distinguished triangle

$$\text{Hom}_{\mathcal{T}}((A, B, \varphi), (A', B', \varphi')) \rightarrow \text{Hom}_{\mathcal{A}}(A, A') \oplus \text{Hom}_{\mathcal{B}}(B, B') \xrightarrow{\phi} \Phi(A, B')$$

where $\phi(\alpha, \beta) = \beta\varphi - \varphi'\alpha$.

More explicitly, $\text{Hom}_{\mathcal{T}}^i((A, B, \varphi), (A', B', \varphi')) = \text{Hom}_{\mathcal{A}}^i(A, A') \oplus \text{Hom}_{\mathcal{B}}^i(B, B') \oplus \Phi^{i-1}(A, B')$ with appropriate differential.

The composition of morphisms in $\mathcal{T}$ is defined as

$$(\alpha, \beta, \psi) \circ (\alpha', \beta', \psi') = (\alpha \circ \alpha', \beta \circ \beta', \psi\alpha' - (-1)^{\deg \beta} \beta\psi').$$

By [KL15, Lemma 4.3] if $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated, so is $\mathcal{T}$.

Note that [HLS16] considered instead two DG algebras A and B , an $A - B$ bimodule F and a new DG algebra

$$C = \begin{pmatrix} A & F \\ 0 & B \end{pmatrix}.$$

The glued category was then the category of DG modules over C .

We have DG functors

$$\begin{aligned} \iota_{\mathcal{A}}: \mathcal{A} &\longrightarrow \mathcal{T}, & \iota_{\mathcal{A}}(A) &= (A, 0, 0), \\ \iota_{\mathcal{B}}: \mathcal{B} &\longrightarrow \mathcal{T}, & \iota_{\mathcal{B}}(B) &= (0, B, 0), \\ \iota_{\mathcal{A}}^*: \mathcal{T} &\longrightarrow \mathcal{A}, & \iota_{\mathcal{A}}^*(A, B, \varphi) &= A, \\ \iota_{\mathcal{B}}^!: \mathcal{T} &\longrightarrow \mathcal{B}, & \iota_{\mathcal{B}}^!(A, B, \varphi) &= B. \end{aligned}$$

An easy calculation shows that $\iota_{\mathcal{A}}^* \dashv \iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}} \dashv \iota_{\mathcal{B}}^!$ are adjoint DG functors:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}((A, B, \varphi), (A', 0, 0)) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, A') \oplus \mathrm{Hom}_{\mathcal{B}}(B, 0) \longrightarrow \Phi(A, 0) \\ \mathrm{Hom}_{\mathcal{T}}((0, B, 0), (A', B', \varphi')) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(0, A') \oplus \mathrm{Hom}_{\mathcal{B}}(B, B') \longrightarrow \Phi(0, B'). \end{aligned}$$

We denote by $\mathcal{A}$ the homotopy category of $\mathcal{A}$, analogously $\mathcal{B} = H^0(\mathcal{B})$ and $\mathcal{T} = H^0(\mathcal{T})$. $i_{\mathcal{A}}: \mathcal{A} \longrightarrow \mathcal{T}$ is the functor of homotopy categories induced by $\iota_{\mathcal{A}}$ and similarly for $i_{\mathcal{B}}$ and all their adjoint functors.

Lemma 3.9. [KL15, Corollary 4.5] *Assume that categories $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated with homotopy categories $\mathcal{A}, \mathcal{B}$. Then the homotopy category $\mathcal{T}$ of $\mathcal{T}$ admits an SOD $\langle \mathcal{A}, \mathcal{B} \rangle$. Moreover, the gluing bifunctor is $[\Phi]: \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathcal{D}(k)$.*

Proof. Functors $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ are clearly fully faithful, hence so are the functors of homotopy categories. We have $\mathrm{Hom}_{\mathcal{T}}(\iota_{\mathcal{B}}(B), \iota_{\mathcal{A}}(A)) = 0$, hence $\mathrm{Hom}_{\mathcal{T}}(\mathcal{B}, \mathcal{A}) = 0$. It remains to check that if both $\iota_{\mathcal{A}}^*(T)$ and $\iota_{\mathcal{B}}^!(T)$ are null-homotopic then so is T . We refer the reader to the proof of [KL15, Corollary 4.5] where an explicit homotopy is given.

The gluing bifunctor is $\mathrm{Hom}_{\mathcal{T}}(i_{\mathcal{A}}(-), i_{\mathcal{B}}(=)[1]) \simeq H^0(\Phi(-, =))$. $\square$

Let us now assume that the bifunctor Φ is given by a gluing functor $F: \mathcal{A} \longrightarrow \mathcal{B}$,

$$(12) \quad \Phi(A, B) = \mathrm{Hom}_{\mathcal{B}}(F(A), B).$$

Then functor $i_{\mathcal{B}}: \mathcal{B} \longrightarrow \mathcal{T}$ of homotopy categories admits the left adjoint:

Lemma 3.10. *Assume that categories $\mathcal{A}, \mathcal{B}$ are pretriangulated and that the bimodule Φ is given by functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ as in (12). Then*

- (a) *Functor $i_{\mathcal{B}}: \mathcal{B} \longrightarrow \mathcal{T}$ admits left adjoint $i_{\mathcal{B}}^*(A, B, f: F(A) \longrightarrow B) = \mathrm{Cone}(f)$.*
- (b) *The homotopy category $\mathcal{T}$ admits an SOD*

$$\mathcal{T} = \langle i_{\mathcal{B}}\mathcal{B}, i_{\mathcal{A}!}\mathcal{A} \rangle$$

where $i_{\mathcal{A}!}: \mathcal{A} \longrightarrow \mathcal{T}$, $i_{\mathcal{A}!}(A) = (A, F(A), \mathrm{Id})$ is the left adjoint of $i_{\mathcal{A}}^*$.

Proof. First, we check that $i_{\mathcal{B}}^* \dashv i_{\mathcal{B}}$:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}((A, B, f), (0, B', 0)) &= \\ \mathrm{Cone}(\mathrm{Hom}_{\mathcal{B}}(B, B')) &\xrightarrow{(-) \circ f} \mathrm{Hom}_{\mathcal{B}}(F(A), B')[-1] \simeq \\ \mathrm{Hom}_{\mathcal{B}}(\mathrm{Cone}(f), B'). \end{aligned}$$

Since $\mathcal{B} \subset \mathcal{T}$ is admissible, category $\mathcal{T}$ admits an SOD $\langle \mathcal{B}, i_{\mathcal{A}!}\mathcal{A} \rangle$, where $i_{\mathcal{A}!}$ is the left adjoint to $i_{\mathcal{A}}$, see Lemma 2.19. It remains to check that, indeed, $\mathrm{Hom}_{\mathcal{T}}(i_{\mathcal{A}!}(A), T) \simeq \mathrm{Hom}_{\mathcal{A}}(A, i_{\mathcal{A}}^*(T))$, for $i_{\mathcal{A}!}$ as defined above:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}((A, F(A), \mathrm{Id}), (A', B', f')) &\simeq \\ \mathrm{Cone}(\mathrm{Hom}_{\mathcal{A}}(A, A') \oplus \mathrm{Hom}_{\mathcal{B}}(F(A), B')) &\xrightarrow{\psi} \mathrm{Hom}_{\mathcal{B}}(F(A), B')[-1] \\ \mathrm{Hom}_{\mathcal{A}}(A, A'). \end{aligned}$$

Indeed, $\psi(\alpha, \beta) = \beta - f'\alpha$ is the identity morphism when restricted to the direct summand $\mathrm{Hom}_{\mathcal{B}}(F(A), B')$. $\square$

Note that $i_{\mathcal{A}!} \dashv i_{\mathcal{A}}^*$ on the level of homotopy categories only.

Let us now assume that the bifunctor Φ is given by the functor $R: \mathcal{B} \rightarrow \mathcal{A}$:

$$(13) \quad \Phi(A, B) = \text{Hom}_{\mathcal{A}}(A, R(B)).$$

Then $i_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{T}$ admits the right adjoint.

Lemma 3.11. *Assume that categories $\mathcal{A}$, $\mathcal{B}$ are pretriangulated and that the bimodule Φ is given by $R: \mathcal{B} \rightarrow \mathcal{A}$ as in (13). Then*

- (a) *Functor $i_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{T}$ admits right adjoint $i_{\mathcal{A}}^!(A, B, g: A \rightarrow R(B)) = \text{Cone}(g)[-1]$.*
- (b) *The homotopy category $\mathcal{T}$ admits an SOD*

$$\mathcal{T} = \langle i_{\mathcal{B}^?} \mathcal{B}, i_{\mathcal{A}} \mathcal{A} \rangle,$$

where $i_{\mathcal{B}^?}: \mathcal{B} \rightarrow \mathcal{T}$, $i_{\mathcal{B}^?}(B) = (R(B), B, \text{Id}: R(B) \rightarrow R(B))$ is the right adjoint of $i_{\mathcal{B}}^!$.

3.4. Spherical functors and 4-periodic decompositions. Consider a triangulated category $\mathcal{T}$ with an SOD $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$. Assume further that $\mathcal{A}$, $\mathcal{B}$ and all their iterated mutations are admissible subcategories in $\mathcal{T}$. Then $\mathbb{Z}$, viewed as the braid group on two strands, acts on the set of SODs for $\mathcal{T}$ via

$$\langle \mathcal{T}_1, \mathcal{T}_2 \rangle \xrightarrow{1} \langle \mathcal{T}_2, R_{\mathcal{T}_2} \mathcal{T}_1 \rangle.$$

We say that the SOD $\langle \mathcal{A}, \mathcal{B} \rangle$ is n -periodic if the $\mathbb{Z}$ action factors via $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$, [BD24]. In particular, (10) is an example of a 4-periodic SOD.

We check that if functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is spherical and $\Phi_F(A, B) = \text{Hom}(F(A), B)$, then the homotopy category of $\mathcal{T} = \mathcal{A} \times_{\Phi_F} \mathcal{B}$ is 4-periodic. Further, we show that the gluing functor for a 4-periodic SOD in a triangulated category behaves like a spherical functor, i.e. it admits canonical twist and cotwist which are equivalences.

Theorem 3.12. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a spherical functor (in the sense of Anno Logvinenko) of pretriangulated DG categories. Then the homotopy category of $\mathcal{T}$ glued via the bimodule $\Phi(-, =) = \text{Hom}_{\mathcal{B}}(F(-), =)$ admits four-periodic semi-orthogonal decomposition:*

$$\mathcal{T} = \langle i_{\mathcal{B}^?} \mathcal{B}, i_{\mathcal{A}} \mathcal{A} \rangle = \langle i_{\mathcal{A}} \mathcal{A}, i_{\mathcal{B}} \mathcal{B} \rangle = \langle i_{\mathcal{B}} \mathcal{B}, i_{\mathcal{A}!} \mathcal{A} \rangle = \langle i_{\mathcal{A}!} \mathcal{A}, i_{\mathcal{B}^?} \mathcal{B} \rangle.$$

Proof. The first three decompositions follow from Lemmas 3.9, 3.10 and 3.11. It remains to check that $\mathcal{T} = \langle i_{\mathcal{A}!} \mathcal{A}, i_{\mathcal{B}^?} \mathcal{B} \rangle$.

First, we check that $\text{Hom}_{\mathcal{T}}(i_{\mathcal{B}^?}(B), i_{\mathcal{A}!}(A)) = 0$ for any $A \in \mathcal{A}$, $B \in \mathcal{B}$:

$$\begin{aligned} & \text{Hom}_{\mathcal{T}}((R(B), B, \text{Id}: R(B) \rightarrow R(B)), (A, F(A), \eta_A: A \rightarrow RF(A))) \simeq \\ & \text{Cone}(\text{Hom}_{\mathcal{A}}(R(B), A) \oplus \text{Hom}_{\mathcal{B}}(B, F(A)) \xrightarrow{\Upsilon} \text{Hom}_{\mathcal{A}}(R(B), F(A)))[-1], \end{aligned}$$

where $\Upsilon(a, b) = R(b) - \eta_A \circ a$.

Consider the following diagram with distinguished rows and columns

$$\begin{array}{ccccc} 0 & \longrightarrow & \text{Hom}_{\mathcal{A}}(R(B), A) \oplus \text{Hom}_{\mathcal{B}}(B, F(A)) & \longrightarrow & \text{Hom}_{\mathcal{A}}(R(B), RF(A)) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Hom}_{\mathcal{A}}(R(B), C(A)) & \xrightarrow{\theta_A \circ -} & \text{Hom}_{\mathcal{A}}(R(B), A) & \longrightarrow & \text{Hom}_{\mathcal{A}}(R(B), RF(A)) \\ \uparrow \psi & & \uparrow 0 & & \uparrow \\ \text{Hom}_{\mathcal{B}}(B, F(A)[-1]) & \longrightarrow & \text{Hom}_{\mathcal{B}}(B, F(A)[-1]) & \longrightarrow & 0 \end{array}$$

where $C: \mathcal{A} \rightarrow \mathcal{A}$ is the cotwist, $\theta: C \rightarrow \text{Id}_{\mathcal{A}}$ is the natural transformation in the definition of C .

Morphism ψ is the isomorphism:

$$\text{Hom}_{\mathcal{B}}(B[1], F(A)) \simeq \text{Hom}_{\mathcal{A}}(L(B)[1], A) \simeq \text{Hom}_{\mathcal{A}}(CL(B)[1], C(A)) \simeq \text{Hom}_{\mathcal{A}}(R(B), C(A)),$$

where the last isomorphism is given by the isomorphism $CL[1] \simeq R$ in the definition of a spherical functor. The isomorphism above maps $b: B[1] \rightarrow F(A)$ to

$$R(B) \xrightarrow{R(\eta_B^L)} RFL(B) \xrightarrow{\beta_{LB}} CL(B)[1] \xrightarrow{CL(b)} CLF(A) \xrightarrow{C(\varepsilon_A^L)} C(A).$$

The composition of the above morphism with θ_A is indeed zero, as the following diagram commutes:

$$\begin{array}{ccccccc} R(B) & \xrightarrow{R(\eta_B^L)} & RFL(B) & \xrightarrow{\beta_{LB}} & CL(B)[1] & \xrightarrow{CL(b)} & CLF(A) \xrightarrow{C(\varepsilon_A^L)} C(A) \\ & & \searrow 0 & & \downarrow \theta_{L(B)} & & \downarrow \theta_{LF(A)} \\ & & & & L(B)[1] & \xrightarrow{L(b)} & LF(A) \xrightarrow{\varepsilon^L(A)} A \\ & & & & & & \downarrow \theta_A \end{array}$$

Hence, $\text{Hom}_{\mathcal{T}}(i_{\mathcal{B}^?}(B), i_{\mathcal{A}^!}(A))$ is the cone of an isomorphism, i.e. it is homotopic to zero.

It remains to check that categories $i_{\mathcal{B}^?}\mathcal{B}$ and $i_{\mathcal{A}^!}\mathcal{A}$ span $\mathcal{T}$. To this end we show that any T in ${}^\perp(i_{\mathcal{A}^!}\mathcal{A})$ lies in the essential image of $i_{\mathcal{B}^?}\mathcal{B}$.

Any $T = (A, B, f: A \rightarrow R(B))$ fits into a distinguished triangle

$$(0, B, 0) \rightarrow T \rightarrow (A, 0, 0) \rightarrow (0, B, 0)[1],$$

where the connecting morphism correspond to f via the isomorphism

$$\text{Hom}_{\mathcal{T}}(\iota_{\mathcal{A}}(A), \iota_{\mathcal{B}}(B)[1]) \simeq \text{Hom}_{\mathcal{A}}(A, \iota_{\mathcal{A}}^! \iota_{\mathcal{B}}(B[1])) \simeq \text{Hom}_{\mathcal{A}}(A, R(B)).$$

We shall argue that if $\text{Hom}_{\mathcal{T}}(T, i_{\mathcal{A}^!}(A')) = 0$ for any $A' \in \mathcal{A}$, then f is an isomorphism, i.e. $T \in i_{\mathcal{B}^?}\mathcal{B}$. We note that $\text{Hom}_{\mathcal{T}}(T, i_{\mathcal{A}^!}(A')) = 0$ if and only if the morphism $\text{Hom}_{\mathcal{T}}(i_{\mathcal{B}}(B)[1], i_{\mathcal{A}^!}(A')) \rightarrow \text{Hom}_{\mathcal{T}}(i_{\mathcal{A}}(A), i_{\mathcal{A}^!}(A'))$ is an isomorphism.

Note that

$$\text{Hom}_{\mathcal{T}}(\iota_{\mathcal{A}}(A), \iota_{\mathcal{A}}(A')) \simeq \text{Cone}(\text{Hom}_{\mathcal{A}}(A, A) \rightarrow \text{Hom}_{\mathcal{A}}(A, RF(A))[-1],$$

hence

$$\text{Hom}_{\mathcal{T}}(i_{\mathcal{A}}(A), i_{\mathcal{A}^!}(A')) \simeq \text{Hom}_{\mathcal{A}}(A, C(A')).$$

Further,

$$\text{Hom}_{\mathcal{T}}(\iota_{\mathcal{B}}(B), \iota_{\mathcal{A}}(A')) \simeq \text{Cone}(\text{Hom}_{\mathcal{B}}(B, F(A)) \rightarrow 0)[-1],$$

hence

$$\text{Hom}_{\mathcal{T}}(i_{\mathcal{B}}(B), i_{\mathcal{A}^!}(A')) \simeq \text{Hom}_{\mathcal{B}}(B, F(A')) \simeq \text{Hom}_{\mathcal{A}}(L(B), A') \simeq \text{Hom}_{\mathcal{A}}(CL(B), C(A')),$$

as C is an equivalence. It follows that $T \in {}^\perp i_{\mathcal{A}^!}\mathcal{A}$ if and only if morphism $\text{Hom}_{\mathcal{A}}(CL(B)[1], C(A')) \rightarrow \text{Hom}_{\mathcal{A}}(A, C(A'))$ induced by $f: A \rightarrow R(B)$ is an isomorphism for any $A' \in \mathcal{A}$. As F is spherical, $R(B) \simeq CL(B)[1]$. One checks that $T = (A, B, f: A \rightarrow R(B))$ is an object in ${}^\perp i_{\mathcal{A}^!}\mathcal{A}$ if and only if $\text{Hom}(R(B), C(A')) \xrightarrow{-\circ f} \text{Hom}(A, C(A'))$ is an isomorphism, for any $A' \in \mathcal{A}$ i.e. if and only if f is an isomorphism (as $C: \mathcal{A} \rightarrow \mathcal{A}$ is an equivalence). $\square$

Note that the inverse of Theorem 3.12 works for an arbitrary triangulated category $\mathcal{T}$:

Theorem 3.13. *Consider a triangulated category $\mathcal{T}$ with a 4-periodic SOD (10). Then*

- (a) *the gluing functor $i_{\mathcal{B}}^* i_{\mathcal{A}}$ is spherical, i.e. the (canonically defined) twist and cotwist are equivalences. Moreover, $T \simeq L_{\mathcal{C}} L_{\mathcal{A}}$, $C \simeq R_{\mathcal{D}} R_{\mathcal{B}}$, $T' \simeq R_{\mathcal{A}} R_{\mathcal{C}}$ and $C' \simeq L_{\mathcal{B}} L_{\mathcal{D}}$ are compositions of mutation functors.*
- (b) *the co-gluing functor $i_{\mathcal{A}}^! i_{\mathcal{B}}$ is spherical, i.e. the (canonically defined) twist and cotwist are equivalences. Moreover, $T \simeq L_{\mathcal{B}} L_{\mathcal{D}}$, $C \simeq R_{\mathcal{A}} R_{\mathcal{C}}$, $T' \simeq R_{\mathcal{D}} R_{\mathcal{B}}$ and $C' \simeq L_{\mathcal{C}} L_{\mathcal{A}}$ are compositions of mutation functors.*

Proof. As $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are admissible subcategories of $\mathcal{T}$, functors $i_{\mathcal{A}}$ and $i_{\mathcal{B}}^*$ admit both adjoints, see Lemma 2.19. In particular $F = i_{\mathcal{B}}^* i_{\mathcal{A}}$ admits left and right adjoints and $R = i_{\mathcal{A}}^! i_{\mathcal{B}}$. We have functorial distinguished triangles:

$$(14) \quad i_{\mathcal{B}}^* i_{\mathcal{A}} i_{\mathcal{A}}^! i_{\mathcal{B}} \rightarrow \text{Id}_{\mathcal{B}} \rightarrow i_{\mathcal{B}}^* i_{\mathcal{D}} i_{\mathcal{D}}^* i_{\mathcal{B}} \rightarrow i_{\mathcal{B}}^* i_{\mathcal{A}} i_{\mathcal{A}}^! i_{\mathcal{B}}[1],$$

$$(15) \quad i_{\mathcal{A}}^! i_{\mathcal{C}} i_{\mathcal{C}}^! i_{\mathcal{A}} \longrightarrow \mathrm{Id}_{\mathcal{A}} \longrightarrow i_{\mathcal{A}}^! i_{\mathcal{B}} i_{\mathcal{B}}^* i_{\mathcal{A}} \longrightarrow i_{\mathcal{A}}^! i_{\mathcal{C}} i_{\mathcal{C}}^! i_{\mathcal{A}}[1].$$

Indeed, the first one is the functorial distinguished triangle for the SOD $\langle \mathcal{D}, \mathcal{A} \rangle$ composed with $i_{\mathcal{B}}$ on the right and $i_{\mathcal{B}}^*$ on the left (note that $i_{\mathcal{B}}$ is fully faithful, hence $i_{\mathcal{B}}^* i_{\mathcal{B}} \simeq \mathrm{Id}_{\mathcal{B}}$). Analogously, the second triangle is the functorial distinguished triangle for the SOD $\langle \mathcal{B}, \mathcal{C} \rangle$ composed with $i_{\mathcal{A}}$ on the right and $i_{\mathcal{A}}^!$ on the left.

By definition of mutation functors $T = i_{\mathcal{B}}^* i_{\mathcal{D}} i_{\mathcal{D}}^* i_{\mathcal{B}} = L_{\mathcal{C}} L_{\mathcal{A}}$, $C = i_{\mathcal{A}}^! i_{\mathcal{C}} i_{\mathcal{C}}^! i_{\mathcal{A}} = R_{\mathcal{D}} R_{\mathcal{B}}$. In particular, the twist T and the cotwist C are equivalences.

To calculate the dual twist and dual cotwist for F we compose it with the equivalence $R_{\mathcal{C}} = i_{\mathcal{D}}^! i_{\mathcal{B}}$. We note that $F' = R_{\mathcal{C}} \circ F = i_{\mathcal{D}}^! i_{\mathcal{B}} i_{\mathcal{B}}^* i_{\mathcal{A}}$ is isomorphic to $i_{\mathcal{D}}^! i_{\mathcal{A}}$ as the fiber $i_{\mathcal{D}}^! i_{\mathcal{C}} i_{\mathcal{C}}^! i_{\mathcal{A}}$ of the $i_{\mathcal{B}}^* \dashv i_{\mathcal{B}}$ adjunction unit composed with $i_{\mathcal{A}}$ and $i_{\mathcal{D}}^!$ is isomorphic to zero (because $i_{\mathcal{D}}^! i_{\mathcal{C}} \simeq 0$). Then the left dual L' to F' is $i_{\mathcal{A}}^* i_{\mathcal{D}}$ and the adjunction unit and counit fit into distinguished triangles

$$\begin{aligned} i_{\mathcal{D}}^! i_{\mathcal{B}} i_{\mathcal{B}}^! i_{\mathcal{D}} &\longrightarrow \mathrm{Id} \longrightarrow i_{\mathcal{D}}^! i_{\mathcal{A}} i_{\mathcal{A}}^* i_{\mathcal{D}} = F' L', \\ L' F' &= i_{\mathcal{A}}^* i_{\mathcal{D}} i_{\mathcal{D}}^! i_{\mathcal{A}} \longrightarrow \mathrm{Id} \longrightarrow i_{\mathcal{A}}^* i_{\mathcal{C}} i_{\mathcal{C}}^* i_{\mathcal{A}}. \end{aligned}$$

It follows that the dual twist for F' is $R_{\mathcal{C}} R_{\mathcal{A}} = i_{\mathcal{D}}^! i_{\mathcal{B}} i_{\mathcal{B}}^! i_{\mathcal{D}}: \mathcal{D} \longrightarrow \mathcal{D}$ while the dual cotwist for F' is $L_{\mathcal{B}} L_{\mathcal{D}} = i_{\mathcal{A}}^* i_{\mathcal{C}} i_{\mathcal{C}}^* i_{\mathcal{A}}$. Finally, we get that the dual twist and the dual cotwist for $F = i_{\mathcal{B}}^* i_{\mathcal{A}}$ are $T' = L_{\mathcal{C}} R_{\mathcal{C}} R_{\mathcal{A}} R_{\mathcal{C}} \simeq R_{\mathcal{A}} R_{\mathcal{C}}$ and $C' \simeq L_{\mathcal{B}} L_{\mathcal{D}}$.

In case of the co-gluing functor $G = i_{\mathcal{A}}^! i_{\mathcal{B}}$ the left adjoint $L_G = i_{\mathcal{B}}^* i_{\mathcal{A}}$ is explicit. It follows from (14) that the dual cotwist is $i_{\mathcal{B}}^* i_{\mathcal{D}} i_{\mathcal{D}}^* i_{\mathcal{B}} \simeq L_{\mathcal{C}} L_{\mathcal{A}}$, while (15) implies that the dual twist is $i_{\mathcal{A}}^! i_{\mathcal{C}} i_{\mathcal{C}}^! i_{\mathcal{A}} \simeq R_{\mathcal{D}} R_{\mathcal{B}}$.

To calculate the twist and cotwist we compose G with the mutation functor $L_{\mathcal{B}}$ to get $G' = L_{\mathcal{B}} G = i_{\mathcal{C}}^* i_{\mathcal{A}} i_{\mathcal{A}}^! i_{\mathcal{B}}$. Analogously as before $G' \simeq i_{\mathcal{C}}^* i_{\mathcal{B}}$ as $i_{\mathcal{C}}^* i_{\mathcal{D}} i_{\mathcal{D}}^* i_{\mathcal{B}} \simeq 0$. Then the right adjoint R' to G' is $i_{\mathcal{B}}^! i_{\mathcal{C}}$ and the adjunction unit and counit fit into distinguished triangles

$$\begin{aligned} G' R' &= i_{\mathcal{C}}^* i_{\mathcal{B}} i_{\mathcal{B}}^! i_{\mathcal{C}} \longrightarrow \mathrm{Id} \longrightarrow i_{\mathcal{C}}^* i_{\mathcal{A}} i_{\mathcal{A}}^* i_{\mathcal{C}}, \\ i_{\mathcal{B}}^! i_{\mathcal{D}} i_{\mathcal{D}}^! i_{\mathcal{B}} &\longrightarrow \mathrm{Id} \longrightarrow i_{\mathcal{B}}^! i_{\mathcal{C}} i_{\mathcal{C}}^* i_{\mathcal{B}} = R' G'. \end{aligned}$$

It follows that the twist for G' is $L_{\mathcal{D}} L_{\mathcal{B}} = i_{\mathcal{C}}^* i_{\mathcal{A}} i_{\mathcal{A}}^* i_{\mathcal{C}}$ and the cotwist for G' is $R_{\mathcal{A}} R_{\mathcal{C}} = i_{\mathcal{B}}^! i_{\mathcal{D}} i_{\mathcal{D}}^! i_{\mathcal{B}}$. We conclude that $R_{\mathcal{A}} R_{\mathcal{C}}$ is the cotwist for G , while the twist is $L_{\mathcal{B}} L_{\mathcal{D}} L_{\mathcal{B}} R_{\mathcal{B}} \simeq L_{\mathcal{B}} L_{\mathcal{D}}$. $\square$

3.5. Any autoequivalence is a spherical twist. We recall the argument of Segal [Seg18] that any equivalence $T: \mathcal{B} \longrightarrow \mathcal{B}$ is a spherical twist for an appropriate spherical functor $F: \mathcal{A}_T \longrightarrow \mathcal{B}$.

We need to assume that T and T^{-1} are DG functors $\mathcal{B} \longrightarrow \mathcal{B}$. As one usually considers quasi-functors of DG categories, one might need to replace $\mathcal{B}$ with its cofibrant model [Tab05].

The source category $\mathcal{A}_T$ of the spherical functor is the pretriangulated hull of DG category $\mathcal{A}_T$. Its objects are objects of $\mathcal{B}$, denoted by $j^* B$. We have

$$\mathrm{Hom}_{\mathcal{A}_T}(j^* B_1, j^* B_2) = \mathrm{Hom}_{\mathcal{B}}(B_1, B_2) \oplus \mathrm{Hom}_{\mathcal{B}}(B_1, T^{-1}(B_2)[1]),$$

with composition $(f, f') \circ (g, g') = (f \circ g, f' \circ g + T^{-1}(f) \circ g')$. The natural faithful functor $L: \mathcal{B} \longrightarrow \mathcal{A}_T$ admits right adjoint F ,

$$F(j^* B) = B \oplus T^{-1}(B), \quad F(f, f') = \begin{pmatrix} f & f' \\ 0 & T^{-1}f \end{pmatrix}: B \oplus T^{-1}B \longrightarrow B' \oplus T^{-1}(B').$$

Then the right adjoint to F is $L \circ T[-1]$, hence F is spherical as conditions (a) and (d) are satisfied.

4. EXAMPLES OF PERIODIC SODS

We discuss two geometric examples of 4-periodic SODs.

4.1. Periodic SODs for an effective Cartier divisor. Let us discuss now the spherical functor $i_*: \mathcal{D}^b(D) \rightarrow \mathcal{D}^b(X)$ for an effective Cartier divisor $D \subset X$, where $\mathcal{D}^b(X) = \mathcal{D}^b(\text{Coh}(X))$ denotes the bounded derived category of coherent sheaves on a noetherian scheme X .

In the case when D and X are smooth projective, i.e. when Fourier-Mukai kernels allow to define spherical functors, the functor i_* has been shown to be spherical by Kuznetsov [Kuz19]. In joint work with Donovan, [BD24] we proved that the functor is triangle spherical under much weaker assumptions.

We consider a reduced separated noetherian scheme X over an algebraically closed field k of characteristic zero. On X we consider an effective Cartier divisor D given by the data $\{(U_i, f_i)\}$ of an open cover of X and regular functions. To this data one assigns the n 'th root stack $\sqrt[n]{X/D}$. Variation of GIT studied by [HL15] yields a $2n$ -periodic SOD of $\mathcal{D}^b(\sqrt[n]{X/D})$.

The root stack can be defined as the fiber product

$$\begin{array}{ccc} \sqrt[n]{X/D} & \longrightarrow & [\mathbb{A}_k^1/\mathbb{G}_m] \\ \downarrow & & \downarrow e_n \\ X & \xrightarrow{\delta} & [\mathbb{A}_k^1/\mathbb{G}_m] \end{array}$$

where $[\mathbb{A}_k^1/\mathbb{G}_m]$ is the quotient stack which represents the functor that to a scheme B assigns the groupoid of generalised Cartier divisors $(L, \rho: L \rightarrow \mathcal{O}_B)$, δ is the morphism given by

$$(\mathcal{O}_X(-D), s(-D): \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X)$$

and e_n is the morphism induced by taking the n 'th power $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$.

One can also define $\sqrt[n]{X/D}$ using the presentation (U_i, f_i) of D . More precisely, let T denote the total space of $\mathcal{O}_X(D)$ and T° the total space of the associated $\mathbb{G}_m$ -bundle, i.e. T with the image of the zero section removed. Over U_i let u_i be the local coordinate on T° and let z be a coordinate on $\mathbb{A}_k^1$. Consider $\mathcal{V}_i = \{u_i z^n = f_i\} \subset T^\circ|_{U_i} \times \mathbb{A}_k^1$. The schemes $\mathcal{V}_i$ glue to a scheme $\mathcal{V}$ with a canonical projection $p: \mathcal{V} \rightarrow X$.

We consider $\mathbb{G}_m$ action on $\mathcal{V}$ defined locally via

$$\lambda \cdot (u_i, z) = (\lambda^{-n} u_i, \lambda z).$$

Then

$$\sqrt[n]{X/D} = [\mathcal{V}/\mathbb{G}_m].$$

The periodic SOD for $\sqrt[n]{X/D}$ is induced via variation of GIT from the action of $\mathbb{G}_m$ on a bigger space, namely on $T \times \mathbb{A}_k^1$. We consider the fiberwise $\mathbb{G}_m$ action with weights $(-n, 1)$. The action restricts to the hypersurface $\mathcal{X}$ locally given by the equation $u_i z^n = f_i$.

Proposition 4.1. [BD24, Proposition 2.9] *Consider $\mathcal{X}$ with the above $\mathbb{G}_m$ -action. The associated GIT quotients are as follows:*

- (+) *We have $[\mathcal{X}^+/\mathbb{G}_m] = \sqrt[n]{X/D}$. The unstable locus $S^+ = \{u = 0\}$ is isomorphic to the total space of $\mathcal{O}_D$ with fiberwise $\mathbb{G}_m$ -action of weight 1.*
- (-) *We have $[\mathcal{X}^-/\mathbb{G}_m] \simeq X$. The unstable locus $S^- = \{z = 0\}$ is isomorphic to the total space $\mathcal{O}_D(-D)$ with fiberwise $\mathbb{G}_m$ -action of weight $-n$.*

The fixed locus $Z = \{u = z = 0\}$ is isomorphic to D .

Recall that, for a scheme $\mathcal{X}$ with $\mathbb{G}_m$ -action and fixed locus $Z \subset X$, the unstable loci are

$$S^+ = \{x \in \mathcal{X} \mid \lim_{\lambda \rightarrow 0} \lambda \cdot x \in Z\}, \quad S^- = \{x \in \mathcal{X} \mid \lim_{\lambda^{-1} \rightarrow 0} \lambda \cdot x \in Z\}.$$

The semistable loci are $\mathcal{X}^\pm = \mathcal{X} \setminus S^\pm$.

Sketch of proof of Proposition 4.1. We can consider the local $\mathbb{G}_m$ action over U_i . Then it suffices to study the action on $\mathbb{A}_k^2$ with weights $(-n, 1)$. The fixed loci is $(0, 0)$, the unstable loci are $\{u_i = 0\}$ and $\{z = 0\}$. One quickly verifies that $Z = \{u_i = z = 0\}$ is the zero set of f_i , i.e. $D \cap U_i$. The description of $[\mathcal{X}^+/\mathbb{G}_m]$ and Z follows.

Note that $\mathbb{G}_m$ acts freely on the total space of $\mathcal{O}_X$ with the zero section removed (as $\mathbb{G}_m$ acts freely on $\mathbb{A}_k^1 \setminus \{0\}$). It follows that $\mathcal{X}^-$ is a $\mathbb{G}_m$ -torsor over X and $[\mathcal{X}^-/\mathbb{G}_m] \simeq X$. $\square$

Proposition 4.1 allows to apply the methods of [HL15]. More precisely, one checks that

- $\pi_{\pm}: S_{\pm} \rightarrow Z$ are locally trivial bundles of affine spaces,
- inclusions $S_{\pm} \rightarrow \mathcal{X}$ are regular embeddings, and
- $\det \mathcal{N}_{S_{\pm}|\mathcal{X}}^{\mp 1}|_Z$ has positive $\mathbb{G}_m$ weight $\eta_{\pm}$.

One considers subcategories of $\mathcal{D}^b(X/\mathbb{G}_m)$

$$\mathcal{C}_{[\omega, \omega + \eta_{\pm}]} = \{E \in \mathcal{D}^b(\mathcal{X}/\mathbb{G}_m) \mid H^*(i_Z^* E) \text{ has weights in } [\omega, \omega + \eta_{\pm}]\},$$

$$\mathcal{A}_{\omega} = \{E \in \mathcal{D}^b(\mathcal{X}/\mathbb{G}_m) \mid H^*(i_Z^* E) \text{ has weights in } [\omega, \omega + \eta_{\pm}], E \text{ supported on } S^{-}\}$$

where $i_Z: Z \rightarrow \mathcal{X}$ is the closed immersion of the fixed locus.

We have

Proposition 4.2. [HL15] *Under the above assumptions*

- (a) Category $\mathcal{C}_{[\omega, \omega + \eta_{\pm}]}$ is equivalent to $\mathcal{D}^b(\mathcal{X}^{\pm}/\mathbb{G}_m)$.
- (b) Category $\mathcal{C}_{[0, \eta + l]}$ admits SOD's

$$\begin{aligned} \mathcal{C}_{[0, \eta + l]} &= \langle \mathcal{C}_{[0, \eta]}, \mathcal{A}_0, \dots, \mathcal{A}_{l-1} \rangle = \\ &= \langle \mathcal{A}_0, \mathcal{C}_{[1, \eta + 1]}, \mathcal{A}_1, \dots, \mathcal{A}_{l-1} \rangle = \\ &= \dots \\ &= \langle \mathcal{A}_0, \dots, \mathcal{A}_{l-1}, \mathcal{C}_{[l, \eta + l]} \rangle \end{aligned}$$

- (c) Category $\mathcal{A}_{\omega}$ is equivalent to $\mathcal{D}^b(Z)$.

To apply Proposition 4.2 to the study of the root stack we note that the weights are $\eta^+ = n$, $\eta^- = 1$. Moreover, we have the fiber diagram

$$\begin{array}{ccc} \sqrt[n]{\mathcal{O}_D(D)} & \xrightarrow{i} & \sqrt[n]{X/D} \\ \downarrow q & & \downarrow p \\ D & \longrightarrow & X, \end{array}$$

where $\sqrt[n]{\mathcal{O}_D(D)}$ is the root stack of a line bundle (without a section).

We have

Theorem 4.3. [BD24, Theorem 3.7] *The category $\mathcal{D}^b(\sqrt[n]{X/D})$ admits SODs*

$$\begin{aligned} &\langle p^* \mathcal{D}^b(X), i_* q^* \mathcal{D}^b(D), i_* q^* \mathcal{D}^b(D) \otimes \mathcal{O}\langle 1 \rangle, \dots, i_* q^* \mathcal{D}^b(D) \otimes \mathcal{O}\langle n-2 \rangle \rangle \\ &= \langle i_* q^* \mathcal{D}^b(D), p^* \mathcal{D}^b(X) \otimes \mathcal{O}\langle 1 \rangle, i_* q^* \mathcal{D}^b(D) \otimes \mathcal{O}\langle 1 \rangle, \dots, i_* q^* \mathcal{D}^b(D) \otimes \mathcal{O}\langle n-2 \rangle \rangle \\ &= \dots \\ &= \langle i_* q^* \mathcal{D}^b(D), i_* q^* \mathcal{D}^b(D) \otimes \mathcal{O}\langle 1 \rangle, \dots, i_* q^* \mathcal{D}^b(D) \otimes \mathcal{O}\langle n-2 \rangle, p^* \mathcal{D}^b(X) \otimes \mathcal{O}\langle n-1 \rangle \rangle, \end{aligned}$$

where $\mathcal{O}\langle k \rangle$ is a trivial line bundle on $\sqrt[n]{X/D}$ with $\mathbb{G}_m$ -action of weight k .

Finally, one checks that the autoequivalence $- \otimes \mathcal{O}\langle n \rangle$ preserves the above decompositions and it intertwines with equivalences

$$- \otimes \mathcal{O}_X(-D) \quad \text{and} \quad - \otimes \mathcal{O}_D(-D)$$

of $\mathcal{D}^b(X)$ and $\mathcal{D}^b(D)$. Then we get

Theorem 4.4. [BD24, Theorem 4.3] *Consider the category*

$$\mathcal{D} = \langle i_* q^* \mathcal{D}^b(D), i_* q^* \mathcal{D}^b(D) \otimes \mathcal{O}\langle 1 \rangle, \dots, i_* q^* \mathcal{D}^b(D) \otimes \mathcal{O}\langle n-2 \rangle \rangle \subset \mathcal{D}^b(\sqrt[n]{X/D}).$$

Then the SOD

$$\mathcal{D}^b(\sqrt[n]{X/E}) = \langle p^* \mathcal{D}^b(X), \mathcal{D} \rangle$$

is $2n$ -periodic.

We sketch the proof for the case most relevant to these notes.

Proof of the $n = 2$ case. By Theorem 4.3 category $\mathcal{D}^b(\sqrt{X/D})$ admits SODs

$$\langle p^*\mathcal{D}^b(X), i_*q^*\mathcal{D}^b(D) \rangle = \langle i_*q^*\mathcal{D}^b(D), p^*\mathcal{D}^b(X) \otimes \mathcal{O}\langle 1 \rangle \rangle.$$

Tensoring these SODs with $\mathcal{O}\langle 1 \rangle$ yields two more SODs

$$\langle p^*\mathcal{D}^b(X) \otimes \mathcal{O}\langle 1 \rangle, i_*q^*\mathcal{D}^b(D) \otimes \mathcal{O}\langle 1 \rangle \rangle = \langle i_*q^*\mathcal{D}^b(D) \otimes \mathcal{O}\langle 1 \rangle, p^*\mathcal{D}^b(X) \otimes \mathcal{O}\langle 2 \rangle \rangle.$$

Finally, the subcategory $p^*\mathcal{D}^b(X) \otimes \mathcal{O}\langle 2 \rangle$ is equivalent to $p^*(\mathcal{D}^b(X) \otimes \mathcal{O}_X(-D))$, i.e. to $p^*\mathcal{D}^b(X)$. It follows that the SOD $\langle p^*\mathcal{D}^b(X), i_*q^*\mathcal{D}^b(D) \rangle$ is indeed 4-periodic. $\square$

The gluing functor for the decomposition of Theorem 4.4 is best understood as the functor $\mathcal{D} \longrightarrow \mathcal{D}^b(X)$. We have

Proposition 4.5. [BD24, Proposition 4.4] *The co-gluing functor for the decomposition $\langle p^*\mathcal{D}^b(X), \mathcal{D} \rangle$ after restriction to $i_*q^*\mathcal{D}^b(D) \otimes \mathcal{O}\langle k \rangle$ is i_{D*} , for $k = 0$, and 0 for $k \neq 0$, where $i_D: D \longrightarrow X$ is the closed embedding.*

Corollary 4.6. [BD24, Corollary 4.6] *Let $i_D: D \longrightarrow X$ be the inclusion of an effective Cartier divisor in a reduced separated noetherian scheme X . Then the functor $i_D^*: \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(D)$ is triangle spherical. The twist functor is $T(-) = - \otimes \mathcal{O}_D(-D)[2]$ and the cotwist $C = - \otimes \mathcal{O}_X(-D)$.*

Remark 4.7. Semiorthogonal decompositions for the root stack, without the periodicity statement, have previously been obtained by Ishii and Ueda for smooth D and X [IU15, Theorem 1.6], as well as by Bergh, Lunts and Schnuerer for perfect complexes [BLS16, Theorem 4.7]. The $2n$ -periodicity of a semiorthogonal decomposition for a root stack has previously been discussed by Bondal [Bon18], speaking on joint work with Logvinenko.

Remark 4.8. Once one shows that the decomposition in Theorem 4.4 is induced by a decomposition of stable infinity categories, one gets that the gluing functor for this decomposition is $2n$ -spherical in the sense of [DKS23], see Remark 2.22.

4.2. Periodic SODs for flops. Another example of a 4-periodic decomposition appeared in [BB22] as a generalisation of Example 2.3. The general set-up was later established in [Bar20].

Consider a “roof” of triangulated categories

$$\begin{array}{ccc} & \mathcal{A} & \\ \alpha^{+*} \nearrow & & \nwarrow \alpha_*^- \\ \mathcal{B}^+ & & \mathcal{B}^- \\ & \alpha_*^+ \searrow & \nearrow \alpha^{-*} \end{array}$$

such that $\alpha^{\pm*} \dashv \alpha_*^{\pm}$ are pairs of adjoint functors and $\alpha^{\pm*}$ are fully faithful. Let

$$\mathcal{K} = \{A \in \mathcal{A} \mid \alpha_*^+(A) \simeq 0, \alpha_*^-(A) = 0\}$$

be the intersection of kernels of α_*^+ and α_*^- . Denote by $\bar{\alpha}_*^{\pm}: \mathcal{A}/\mathcal{K} \longrightarrow \mathcal{B}^{\pm}$ the induced functors. We also assume that $\mathcal{A}$, $\mathcal{B}^+$ and $\mathcal{B}^-$ are compactly generated triangulated categories admitting arbitrary direct sums.

Theorem 4.9. [Bar20, Theorem 1.0.1] *Assume that*

$$\Phi^+ = \alpha_*^- \circ \alpha^{+*}: \mathcal{B}^+ \longrightarrow \mathcal{B}^- \quad \text{and} \quad \Phi^- = \alpha_*^+ \circ \alpha^{-*}: \mathcal{B}^- \longrightarrow \mathcal{B}^+$$

are equivalences. Then the category $\mathcal{A}/\mathcal{K}$ admits a 4-periodic SOD:

$$\mathcal{A}/\mathcal{K} = \langle \ker \bar{\alpha}_*^-, \mathcal{B}^- \rangle = \langle \mathcal{B}^-, \ker \bar{\alpha}_*^+ \rangle = \langle \ker \bar{\alpha}_*^+, \mathcal{B}^+ \rangle = \langle \mathcal{B}^+, \ker \bar{\alpha}_*^- \rangle.$$

An example of a roof of triangulated categories is given by a diagram of schemes

$$(16) \quad \begin{array}{ccc} & W & \\ p^+ \swarrow & & \searrow p^- \\ X^+ & & X^- \end{array}$$

such that $Rp_*^- \circ Lp^{+*}: \mathcal{D}_{\text{qc}}(X^+) \longrightarrow \mathcal{D}_{\text{qc}}(X^-)$ and $Rp_*^+ \circ Lp^{-*}: \mathcal{D}_{\text{qc}}(X^-) \longrightarrow \mathcal{D}_{\text{qc}}(X^+)$ are equivalences.

Note that in this case Theorem 4.9 gives a spherical functor between the “big” derived categories of quasi-coherent sheaves. To get a spherical functor between bounded derived categories of coherent sheaves one needs more assumptions

Theorem 4.10. [Bar20, Theorem 1.0.2] *Consider diagram (16) which induces a roof of triangulated categories and assume that p^+ and p^- are proper of a finite Tor dimension. Then*

$$\mathcal{D}^b(W)/\mathcal{K}^b = \langle \ker p_*^-, \mathcal{D}^b(X^-) \rangle = \langle \mathcal{D}^b(X^-), \ker p_*^+ \rangle = \langle \ker p_*^+, \mathcal{D}^b(X^+) \rangle = \langle \mathcal{D}^b(X^+), \ker p_*^- \rangle.$$

Another example of a periodic decomposition for “small” derived categories appeared in [BB22]. Consider a flop diagram

$$\begin{array}{ccc} & W & \\ p^+ \swarrow & & \searrow p^- \\ X^+ & & X^- \\ f^+ \searrow & & \swarrow f^- \\ & Y & \end{array}$$

Assume that X^+ , X^- and Y are normal quasi-projective Gorenstein varieties of dimension greater than or equal to three over a field of characteristic zero. Let further morphisms f^+ and f^- be projective and birational with fibers of dimension bounded by one. Assume that $Rf_*^\pm \mathcal{O}_{X^\pm} = \mathcal{O}_Y$ and the exceptional locus of $f^\pm$ in $X^\pm$ is of codimension greater than one. Finally, let Y have canonical hypersurface singularities of multiplicity two and let f^- be a flop of f^+ . Put $W = X^+ \times_Y X^-$.

For the flopping contraction f^+ the abelian null category

$$\mathcal{A}_{f^+} = \{E \in \text{Coh}(X^+) \mid Rf_*^+(E) = 0\}$$

is the heart of a t -structure on $\ker Rf_*^+$.

Theorem 4.11. [BB22, Theorem 5.17] *Under the above assumptions let*

$$\mathcal{K}^b = \{E \in \mathcal{D}^b(W) \mid Rp_*^+(E) \simeq 0 \simeq Rp_*^-(E)\}.$$

Then the quotient category $\mathcal{D}^b(W)/\mathcal{K}^b$ admits a 4-periodic SOD

$$\langle \mathcal{D}^b(\mathcal{A}_{f^+}), \mathcal{D}^b(X^-) \rangle = \langle \mathcal{D}^b(X^-), \mathcal{D}^b(\mathcal{A}_{f^-}) \rangle = \langle \mathcal{D}^b(\mathcal{A}_{f^-}), \mathcal{D}^b(X^+) \rangle = \langle \mathcal{D}^b(X^+), \mathcal{D}^b(\mathcal{A}_{f^+}) \rangle.$$

The co-gluing functor for the decomposition $\langle \mathcal{D}^b(X^+), \mathcal{D}^b(\mathcal{A}_{f^+}) \rangle$ is the derived functor of the embedding $\mathcal{A}_{f^+} \longrightarrow \text{Coh}(X^+)$.

Note that in the case considered in Theorem 4.11 morphisms p^+ and p^- are of infinite Tor-dimension.

Remark 4.12. The category $\mathcal{A}_{f^+}$ has a projective generator $\mathcal{P}^+$. The endomorphism algebra of $\mathcal{P}^+$ is isomorphic to the contraction algebra considered by Donovan and Wemyss in [DW16], see [BB22, Theorem 6.2]. In [DW16] the authors show that the flop-flop equivalence $Rp_*^+ Lp^{-*} Rp_*^- Lp^{+*}$ is the twist around the contraction algebra. Moreover, the contraction algebra is proved to pro-represent the functor of noncommutative deformations of simple objects in $\mathcal{A}_{f^+}$.

In the case of a pagoda flop, [Rei83], the contraction algebra A_{f+} is commutative and Toda [Tod07] proved that the flop-flop equivalence is the twist for an appropriately defined *fat spherical object*, which is equivalent to a spherical functor $\mathcal{D}^b(\text{mod-}A_{f+}) \rightarrow \mathcal{D}^b(X)$.

In [BB25] a theory of non-commutative deformations of objects in abelian categories is developed. In particular, the objects (ind-)representing deformation functors are artinian abelian categories with finitely many isomorphism classes of simple objects. In [BB25] the authors show that the null category $\mathcal{A}_{f+}$ represents the functor of non-commutative deformations of simple objects in $\mathcal{A}_{f+}$.

5. SPHERICAL FUNCTOR FOR A BLOW-UP

In the last lecture we are going to construct a 4-periodic decomposition generalising the spherical functor mentioned in Example 2.12.

The general set-up is the blow-up of a scheme $\mathcal{Y}$ in a regular subscheme Y :

$$\begin{array}{ccc} E & \xrightarrow{i} & \mathcal{X} \\ \downarrow \pi & & \downarrow f \\ Y & \longrightarrow & \mathcal{Y} \end{array}$$

We denote by $E \subset \mathcal{X}$ the exceptional divisor for f . We shall construct a 4-periodic SOD with co-gluing functor $Ri_*L\pi^*: \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(\mathcal{X})$. By the universal property of the blow-up, $E \subset \mathcal{X}$ is a Cartier divisor. Hence, the root stack $\sqrt{\mathcal{X}/E}$ admits a 4-periodic SOD $\langle \mathcal{D}^b(\mathcal{X}), \mathcal{D}^b(E) \rangle$, see Theorem 4.4. On the other hand, as $R\pi_*\mathcal{O}_E = \mathcal{O}_Y$, projection formula yields fully faithfulness of $L\pi^*: \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(E)$. More precisely, we have an SOD

$$\mathcal{D}^b(E) = \langle \ker R\pi_*, L\pi^*\mathcal{D}^b(Y) \rangle.$$

Therefore,

$$\mathcal{D}^b(\sqrt{\mathcal{X}/E}) = \langle \mathcal{D}^b(\mathcal{X}), \ker R\pi_*, L\pi^*\mathcal{D}^b(Y) \rangle.$$

Following [HLS16] we shall discuss when the 4-periodicity of the SOD for $\mathcal{D}^b(\sqrt{\mathcal{X}/E})$ implies 4-periodicity of the induced SOD for

$$\mathcal{D}^b(\sqrt{\mathcal{X}/E}) / \ker R\pi_* = \langle \mathcal{D}^b(\mathcal{X}), \mathcal{D}^b(Y) \rangle.$$

5.1. Periodic decompositions on the quotient category. We rewrite the full proof of [HLS16, Theorem 4.14] in the purely triangulated set-up.

We consider a triangulated category $\mathcal{T}$ with a 4-periodic semiorthogonal decomposition

$$(17) \quad \mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle =$$

$$(18) \quad \langle \mathcal{B}, \mathcal{C} \rangle =$$

$$(19) \quad \langle \mathcal{C}, \mathcal{D} \rangle =$$

$$(20) \quad \langle \mathcal{D}, \mathcal{A} \rangle.$$

Definition 5.1. An SOD $\mathcal{A} = \langle \mathcal{E}, \mathcal{F} \rangle$ is *compatible* with (17)-(20), if

$$(21) \quad \mathcal{A} = \langle \mathcal{E}, \mathcal{F} \rangle = \langle R_{\mathcal{D}}R_{\mathcal{B}}\mathcal{F}, \mathcal{E} \rangle,$$

where $R_{\mathcal{D}}: \mathcal{C} \xrightarrow{\sim} \mathcal{A}$, $R_{\mathcal{B}}: \mathcal{A} \xrightarrow{\sim} \mathcal{C}$ are the equivalences given by right mutations.

From now on we assume that $\mathcal{A}$ admits an SOD (21) compatible with (17)-(20).

Under these assumptions category $\mathcal{C}$ admits semiorthogonal decompositions

$$(22) \quad \mathcal{C} = \langle R_{\mathcal{B}}\mathcal{E}, R_{\mathcal{B}}\mathcal{F} \rangle$$

$$(23) \quad = \langle L_{\mathcal{D}}\mathcal{E}, L_{\mathcal{D}}\mathcal{F} \rangle$$

$$(24) \quad = \langle L_{\mathcal{D}}R_{\mathcal{D}}R_{\mathcal{B}}\mathcal{F}, L_{\mathcal{D}}\mathcal{E} \rangle = \langle R_{\mathcal{B}}\mathcal{F}, L_{\mathcal{D}}\mathcal{E} \rangle.$$

Lemma 5.2. Under the above assumptions subcategories $\mathcal{E}^{\perp}$ and ${}^{\perp}L_{\mathcal{D}}\mathcal{E}$ of $\mathcal{T}$ are equivalent.

Proof. The statement follows from the semiorthogonal decompositions for $\mathcal{T}$:

$$\begin{aligned} (25) \quad & \langle \mathcal{E}, \mathcal{F}, \mathcal{B} \rangle = \\ (26) \quad & \langle \mathcal{B}, R_{\mathcal{B}}\mathcal{E}, R_{\mathcal{B}}\mathcal{F} \rangle = \\ (27) \quad & \langle \mathcal{B}, R_{\mathcal{B}}\mathcal{F}, L_{\mathcal{D}}\mathcal{E} \rangle = \\ (28) \quad & \langle \mathcal{F}, \mathcal{B}, L_{\mathcal{D}}\mathcal{E} \rangle. \end{aligned}$$

Above, the first decomposition follows from (17) and (21), the second decomposition follows from (18) and (22), the third one from (18) and (24) and the last one is obtained from (27) via the left mutation of $R_{\mathcal{B}}\mathcal{F}$ over $\mathcal{F}$. $\square$

Lemma 5.3. *Under the above assumptions subcategories ${}^{\perp}\mathcal{F}$ and $R_{\mathcal{B}}\mathcal{F}^{\perp}$ of $\mathcal{T}$ are equivalent.*

Proof. The statement follows from the semiorthogonal decompositions for $\mathcal{T}$:

$$\begin{aligned} (29) \quad & \langle \mathcal{D}, \mathcal{E}, \mathcal{F} \rangle = \\ (30) \quad & \langle L_{\mathcal{D}}\mathcal{E}, L_{\mathcal{D}}\mathcal{F}, \mathcal{D} \rangle = \\ (31) \quad & \langle R_{\mathcal{B}}\mathcal{F}, L_{\mathcal{D}}\mathcal{E}, \mathcal{D} \rangle = \\ (32) \quad & \langle R_{\mathcal{B}}\mathcal{F}, \mathcal{D}, \mathcal{E} \rangle. \end{aligned}$$

Above, the first decomposition follows from (20) and (21), the second from (19) and (23), the third one from (19) and (24) and the last one is obtained from (31) via the right mutation of $L_{\mathcal{D}}\mathcal{E}$ over $\mathcal{D}$. $\square$

Proposition 5.4. *Consider the subcategory $\mathcal{G} := \mathcal{E}^{\perp} \cap {}^{\perp}\mathcal{F}$ of $\mathcal{T}$. Then category $\mathcal{T}$ admits semiorthogonal decompositions*

$$\begin{aligned} (33) \quad & \mathcal{T} = \langle \mathcal{E}, \mathcal{G}, \mathcal{F} \rangle = \\ (34) \quad & \langle R_{\mathcal{B}}\mathcal{F}, \mathcal{G}, L_{\mathcal{D}}\mathcal{E} \rangle. \end{aligned}$$

Proof. The statement follows from the definition of $\mathcal{G}$ and Lemmas 5.2 and 5.3. $\square$

Theorem 5.5. *Consider a triangulated category $\mathcal{T}$ with a 4-periodic semiorthogonal decomposition (17) - (20). Assume that the category $\mathcal{A}$ admits a semiorthogonal decomposition (21) compatible with the decomposition of $\mathcal{T}$. Then*

(a) *subcategory $\mathcal{E}^{\perp}$ of $\mathcal{T}$ admits a 4-periodic semiorthogonal decomposition*

$$(35) \quad \langle \mathcal{F}, \mathcal{B} \rangle = \langle \mathcal{B}, R_{\mathcal{B}}\mathcal{F} \rangle = \langle R_{\mathcal{B}}\mathcal{F}, \mathcal{G} \rangle = \langle \mathcal{G}, \mathcal{F} \rangle.$$

(b) *subcategory ${}^{\perp}\mathcal{F}$ of $\mathcal{T}$ admits a 4-periodic semiorthogonal decomposition*

$$(36) \quad \langle \mathcal{D}, \mathcal{E} \rangle = \langle \mathcal{E}, \mathcal{G} \rangle = \langle \mathcal{G}, L_{\mathcal{D}}\mathcal{E} \rangle = \langle L_{\mathcal{D}}\mathcal{E}, \mathcal{D} \rangle.$$

(c) *Under the equivalence $R_{\mathcal{C}}: \mathcal{B} \rightarrow \mathcal{D}$, the composition of the twist $L_{R_{\mathcal{B}}\mathcal{F}}L_{\mathcal{F}}: \mathcal{B} \rightarrow \mathcal{B}$ for the periodic SOD (35) with the twist $L_{\mathcal{E}}L_{L_{\mathcal{D}}\mathcal{E}}: \mathcal{D} \rightarrow \mathcal{D}$ for the periodic SOD (36) is isomorphic to the twist $L_{\mathcal{C}}L_{\mathcal{A}}: \mathcal{B} \rightarrow \mathcal{B}$ for the periodic SOD (17) - (20).*

Proof. The first decomposition in (35) follows from (17) and (21). The fourth decomposition follows from (33). The remaining two decompositions in (35) rely on the equality $\mathcal{E}^{\perp} = {}^{\perp}L_{\mathcal{D}}\mathcal{E}$ of subcategories in $\mathcal{T}$ given by Lemma 5.2. More precisely, the second decomposition follows from (27) while the third one from (34).

The first decomposition in (31) follows from (20) and (21) while the second one from (33). The remaining two decompositions rely on the equality ${}^{\perp}\mathcal{F} = R_{\mathcal{B}}\mathcal{F}^{\perp}$ of subcategories in $\mathcal{T}$ given by Lemma 5.3. More precisely, the third one follows from (34) while the fourth one from (31).

To check that the composition of twists is isomorphic to the ‘big’ twist it suffices to check that

$$\mathcal{B} \xrightarrow{L_{\mathcal{F}}} \mathcal{G} \xrightarrow{L_{R_{\mathcal{B}}\mathcal{F}}} \mathcal{B} \xrightarrow{R_{\mathcal{C}}} \mathcal{D} \xrightarrow{L_{L_{\mathcal{D}}\mathcal{E}}} \mathcal{G} \xrightarrow{L_{\mathcal{E}}} \mathcal{D} \xrightarrow{L_{\mathcal{C}}} \mathcal{B}$$

is isomorphic to $L_{\mathcal{C}}L_{\mathcal{A}}$. As $\mathcal{A} = \langle \mathcal{E}, \mathcal{F} \rangle$, $L_{\mathcal{A}} \simeq L_{\mathcal{E}}L_{\mathcal{F}}$. Hence, it suffices to check that $L_{L_{\mathcal{D}}\mathcal{E}} \circ R_{\mathcal{C}} \circ L_{R_{\mathcal{B}}\mathcal{F}}$ is isomorphic to the identity on $\mathcal{G}$. Note that $\mathcal{C} \simeq L_{\mathcal{D}}\mathcal{A}$ admits an SOD (24). Hence, $R_{\mathcal{C}} \simeq R_{L_{\mathcal{D}}\mathcal{E}} \circ R_{R_{\mathcal{B}}\mathcal{F}}$. As the left and right mutation are mutually quasi-inverse, the statement follows. $\square$

5.2. Periodic decomposition for a blow-up. We consider the blow up $f: \mathcal{X} \rightarrow \mathcal{Y}$ of a scheme $\mathcal{Y}$ in a regular subscheme $Y \subset \mathcal{Y}$. We denote by E the exceptional divisor of f and by $\pi: X \rightarrow Y$ the restriction of f to X .

Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{Y}}$ be the ideal sheaf corresponding to $Y \subset \mathcal{Y}$. Then $E = \text{Proj}_{\mathcal{Y}} \bigoplus \mathcal{I}^n / \mathcal{I}^{n+1}$, and π is the canonical morphism $\text{Proj}_{\mathcal{Y}} \bigoplus \mathcal{I}^n / \mathcal{I}^{n+1} \rightarrow Y$, see [GW10, Remark 13.94]. It follows that $R\pi_*\mathcal{O}_E = \mathcal{O}_Y$. Hence, $R\pi_*L\pi^* \simeq \text{Id}$, i.e. $L\pi^*: \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(E)$ is fully faithful and we have a decomposition

$$\mathcal{D}^b(E) = \langle \ker R\pi_*, L\pi^*\mathcal{D}^b(Y) \rangle.$$

Proposition 5.6. *Let $\mathcal{K} = L\pi^*\mathcal{D}^b(Y)^\perp \subset \mathcal{D}^b(E)$. Then the category $\mathcal{D}^b(E)$ admits semiorthogonal decompositions:*

$$(37) \quad \mathcal{D}^b(E) = \langle \mathcal{K} \otimes \omega_\pi, L\pi^*\mathcal{D}^b(Y) \rangle = \langle L\pi^*\mathcal{D}^b(Y), \mathcal{K} \rangle.$$

Proof. Since Y is regular, functor $\pi^!: \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(E)$ is isomorphic to $L\pi^*(-) \otimes \omega_\pi$. Hence, the fully faithful functor $L\pi^*: \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(E)$ admits two adjoint functors:

$$R\pi_*(- \otimes \omega_\pi) \dashv L\pi^*(-) \dashv R\pi_*(-).$$

It follows, that category $\mathcal{D}^b(E)$ admits semiorthogonal decompositions

$$\mathcal{D}^b(E) = \langle \mathcal{M}, L\pi^*\mathcal{D}^b(Y) \rangle = \langle L\pi^*\mathcal{D}^b(Y), \mathcal{K} \rangle,$$

with

$$\begin{aligned} \mathcal{M} &= \{D \in \mathcal{D}^b(E) \mid \text{Hom}(L\pi^*\mathcal{D}^b(Y), D) = 0\} \\ &= \{D \in \mathcal{D}^b(E) \mid R\pi_*(D) = 0\} \\ \mathcal{K} &= \{D \in \mathcal{D}^b(E) \mid \text{Hom}(D, L\pi^*\mathcal{D}^b(Y)) = 0\} \\ &= \{D \in \mathcal{D}^b(E) \mid \text{Hom}(D \otimes \omega_\pi, \pi^!\mathcal{D}^b(Y)) = 0\} \\ &= \{D \in \mathcal{D}^b(E) \mid R\pi_*(D \otimes \omega_\pi) = 0\}. \end{aligned}$$

Clearly, $- \otimes \omega_\pi: \mathcal{K} \rightarrow \mathcal{M}$ is an equivalence. $\square$

Theorem 5.7. *Consider the blow up f of an integral scheme $\mathcal{Y}$ in a regular subscheme Y . Let $\pi: E \rightarrow Y$ denote the restriction of f to the exceptional divisor. Assume that there exists a line bundle $\mathcal{L}$ on Y such that*

$$\omega_\pi = \pi^*(\mathcal{L}) \otimes \mathcal{O}_E(E).$$

Consider the root stack for the effective Cartier divisor $E \subset \mathcal{X}$:

$$\begin{array}{ccc} \sqrt{\mathcal{O}_E(E)} & \xrightarrow{\iota} & \sqrt{\mathcal{X}/E} \\ \downarrow q & & \downarrow p \\ E & \xrightarrow{i} & \mathcal{X} \end{array}$$

Then the SOD

$$\mathcal{D}^b(\sqrt{\mathcal{X}/E}) / \ker R\pi_* = \langle Lp^*\mathcal{D}^b(\mathcal{X}), R\iota_*Lq^*L\pi^*\mathcal{D}^b(Y) \rangle$$

*is 4-periodic with the co-gluing functor $Ri_*L\pi^*: \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(\mathcal{X})$.*

Proof. By [GW10, Corollary 13.97] $\mathcal{X}$ is integral. By the universal property of the blow-up $E \subset \mathcal{X}$ is an effective Cartier divisor. Hence, we can apply Theorem 4.4 and Proposition 4.5 to conclude that the SOD $\mathcal{D}^b(\sqrt{\mathcal{X}/E}) = \langle p^*\mathcal{D}^b(\mathcal{X}), \iota_*q^*\mathcal{D}^b(E) \rangle$ is 4-periodic with gluing functor $i^*: \mathcal{D}^b(\mathcal{X}) \rightarrow \mathcal{D}^b(E)$. By Corollary 4.6, $T(-) = - \otimes \mathcal{O}_E(-E)[2]$ and $C(-) = - \otimes \mathcal{O}_{\mathcal{X}}(-E)$.

Consider the 4-periodic decomposition

$$\mathcal{D}^b(\sqrt{\mathcal{X}/E}) = \langle \iota_* q^* \mathcal{D}^b(E), p^* \mathcal{D}^b(\mathcal{X}) \otimes \mathcal{O}\langle 1 \rangle \rangle.$$

It follows from Theorem 3.13 and Corollary 4.6 that the double right mutation on $\mathcal{D}^b(E)$ is $T^{-1}(-) = - \otimes \mathcal{O}_E(E)[-2]$. We conclude that the decomposition $\mathcal{D}^b(E) = \langle L\pi^*(Y), \mathcal{K} \rangle$ is compatible with the periodic SOD for the root stack $\sqrt{\mathcal{X}/E}$. Indeed, by assumption on ω_π and (37), we have

$$\mathcal{D}^b(E) = \langle \mathcal{K} \otimes \omega_\pi, L\pi^* \mathcal{D}^b(Y) \rangle = \langle \mathcal{K} \otimes \pi^*(\mathcal{L}) \otimes \mathcal{O}_E(E), L\pi^* \mathcal{D}^b(Y) \rangle.$$

The automorphism $-\otimes \pi^*(\mathcal{L}^{-1})$ of $\mathcal{D}^b(E)$ yields an SOD

$$\begin{aligned} \mathcal{D}^b(E) &= \langle \mathcal{K} \otimes \mathcal{O}_E(E), L\pi^* \mathcal{D}^b(Y) \otimes L\pi^*(\mathcal{L}^{-1}) \rangle \\ &= \langle \mathcal{K} \otimes \mathcal{O}_E(E), L\pi^*(\mathcal{D}^b(Y) \otimes \mathcal{L}^{-1}) \rangle \\ &= \langle \mathcal{K} \otimes \mathcal{O}_E(E)[-2], L\pi^* \mathcal{D}^b(Y) \rangle \\ &= \langle T^{-1}(\mathcal{K}), L\pi^* \mathcal{D}^b(Y) \rangle, \end{aligned}$$

as needed. By Theorem 5.5 the decomposition

$$\mathcal{D}^b(\sqrt{\mathcal{X}/E})/\mathcal{K} = \langle q^* \iota_* \pi^* \mathcal{D}^b(Y), p^* \mathcal{D}^b(\mathcal{X}) \otimes \mathcal{O}\langle 1 \rangle \rangle$$

is 4-periodic. The mutation gives an SOD

$$(38) \quad \langle p^* \mathcal{D}^b(\mathcal{X}), q^* \iota_* \pi^* \mathcal{D}^b(Y) \rangle.$$

It follows from Corollary 4.6 that the co-gluing functor for this decomposition (right adjoint to the gluing functor, Remark 2.21) is $i_*: \mathcal{D}^b(E) \rightarrow \mathcal{D}^b(\mathcal{X})$. Hence, the co-gluing functor for the decomposition (38) is $i_* \pi^*: \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(\mathcal{X})$. $\square$

5.3. More examples of spherical functors. In [Don24] Donovan considers yet another spherical functor related to blow-ups. The construction requires a smooth subscheme $S \subset Y$ of constant codimension in a smooth equidimensional scheme $\mathcal{Y}$. Assume further that $i: Y \rightarrow \mathcal{Y}$ is a hypersurface containing S such that the blow-up $f: X \rightarrow Y$ in Z is crepant, i.e. $\omega_X = f^* \omega_Y$. Denote by $g: X \rightarrow \mathcal{Y}$ the composition $X \xrightarrow{f} Y \xrightarrow{i} \mathcal{Y}$. Then the functor $Rg_*: X \rightarrow \mathcal{Y}$ is spherical in the sense of Kuznetsov, [Don24, Theorem A].

Analogously to Theorem 5.7 one checks that the 4-periodic SOD with (co-)gluing functor Rg_* is an SOD on the quotient of $\mathcal{D}^b(\sqrt{\mathcal{X}/X})$ by the kernel of the pushforward functor along the blow-up $h: \mathcal{X} \rightarrow \mathcal{Y}$ of $\mathcal{Y}$ along Z .

Many examples of spherical objects and functors related to flops, generalising Example 2.5, have been studied by Donovan, himself and with coauthors. Spherical objects for higher dimensional standard flops are discussed in [ADM19]. Grassmannian flops, associated spherical functors, spherical twists and more general autoequivalences are discussed in [DS14] and [Don13].

Addington in [Add16] discusses yet another spherical functor. Namely, let S be a K3 surface and let $\mathcal{U}$ be the universal ideal sheaf on the product of S with the Hilbert scheme $S^{[2]}$ of subschemes of length two on S . Then the Fourier Mukai transform $F: \mathcal{D}^b(S) \rightarrow \mathcal{D}^b(S^{[2]})$ with kernel $\mathcal{U}$ is spherical.

In fact, in this example the composition RF splits into $\text{Id}_{\mathcal{D}^b(S)}$ and $[-2]: \mathcal{D}^b(S) \rightarrow \mathcal{D}^b(S)$. More generally, for the Fourier Mukai transform $\tilde{F}: \mathcal{D}^b(S) \rightarrow \mathcal{D}^b(S^{[n]})$ given by the universal ideal sheaf on the product of S and the Hilbert scheme on n points of S , the composition of $\tilde{F}$ with its right adjoint splits into

$$\text{Id}_{\mathcal{D}^b(S)} \oplus [-2] \oplus \dots \oplus [-2n + 2],$$

see [Add16, Theorem 2]. This led Addington to the definition of (split) $\mathbb{P}^n$ -functors.

$\mathbb{P}^n$ -objects were introduced in [HT06]. These are objects whose Ext-algebra is isomorphic to the cohomology of projective space $\mathbb{P}^n$. A $\mathbb{P}^n$ -object $E \in \mathcal{D}^b(X)$ yields an autoequivalence of $\mathcal{D}^b(X)$. To define it one needs to consider the generator $h: E[-2] \rightarrow E$ of the graded ring

$\text{Ext}^*(E, E)$. It gives $H = h^* \boxtimes \text{Id} - \text{Id} \boxtimes h: E^* \boxtimes E \longrightarrow \mathcal{O}_\Delta$. Let $\mathcal{H}$ be its cone. Then the trace morphism $E^* \boxtimes E \longrightarrow \mathcal{O}_\Delta$ factorizes via $\mathcal{H}$ yielding $t: \mathcal{H} \longrightarrow \mathcal{O}_\Delta$. The Fourier Mukai transform with kernel $\text{Cone}(t)$ is the autoequivalence induced by E . If E is a $\mathbb{P}^1$ object, i.e. if it is 2-spherical then the $\mathbb{P}^n$ -twist is the square of the spherical twist [HT06, Proposition 2.9].

5.4. Where to go from here. In [AL25] Anno and Logvinenko introduced the notion of DG $\mathbb{P}^n$ -functors. Unlike in the example considered by Addington [Add16] the composition of a $\mathbb{P}^n$ -functor F with its right adjoint R is not assumed to split into a direct sum of powers of an autoequivalence H ; this condition is replaced by a condition regarding a degree n cyclic coextension of the identity functor by H .

It would be interesting to find a purely triangulated definition of $\mathbb{P}^n$ -functors in the language of periodicity of SODs, generalising the work [HLS16] of Halpern-Leistner and Shipman.

A “geometric” family of spherical functors is discussed in Example 2.13. They are the pushforward functors along double covers. A natural question to ask is whether there is a triangulated category of a “geometric” origin admitting a 4-periodic SOD with appropriate (co-)gluing functor.

Seidel and Thomas in [ST01] discussed a relation between spherical and exceptional objects. They considered a simple morphism $f: X \longrightarrow Y$ of smooth projective varieties, i.e. a morphism of codimension c whose source is a Calabi-Yau variety, i.e. $\omega_X \simeq \mathcal{O}_X$, and such that $Rf_*\mathcal{O}_X$ fits into a distinguished triangle

$$\mathcal{O}_Y \longrightarrow Rf_*\mathcal{O}_X \longrightarrow \omega_Y[-c].$$

They proved that if f is simple and $E \in \mathcal{D}^b(Y)$ is exceptional then $Lf^*E \in \mathcal{D}^b(X)$ is spherical. More recently, Arai [Ara23] studied objects $E \in \mathcal{D}^b(X_0)$ whose pushforward along an appropriate $i: X_0 \longrightarrow X$ is spherical. Here, i is an inclusion of a fiber of a flat family. The author proved that if i_*E is spherical with a spherical twist T_E then there exists an autoequivalence H of $\mathcal{D}^b(X_0)$ such that $i_* \circ H \simeq T_E \circ i_*$. It might be interesting to try to generalise these results to spherical functors, understood in one of the possible ways discussed in these notes.

In the original paper [ST01] introducing spherical objects, an A_n configuration of spherical objects was considered. This is a collection of spherical objects behaving like the collection $\{\mathcal{O}_{C_i}\}$, for a chain $C_1, \dots, C_n$ of -2 -curves in a smooth surface, see Example 2.3. Seidel and Thomas proved that the spherical twists related to an A_n configuration of spherical objects in $\mathcal{D}^b(X)$ yield an embedding of the braid group B_n on n strands into the group of derived equivalences of X , as soon as $\dim X \geq 2$, [ST01, Theorem 1.3]. This result was generalised by Anno and Logvinenko for the case of spherical DG functors [AL17, Theorem 1.2] where the authors defined an A_n configuration of spherical functors $F_i: \mathcal{A}_i \longrightarrow \mathcal{B}$ and proved that it yields B_n -action on $\mathcal{B}$. One might try to describe this braid group action from the point of view of SODs. Finally, complexes and, more generally, sequences of spherical functors should yield examples of SODs related to schobers, discussed in a different series of lectures.

REFERENCES

- [Add16] N. Addington. New derived symmetries of some hyperkähler varieties. *Algebr. Geom.*, 3(2):223–260, 2016.
- [ADM19] N. Addington, W. Donovan, and C. Meachan. Mukai flops and $\mathbb{P}$ -twists. *J. Reine Angew. Math.*, 748:227–240, 2019.
- [AL17] R. Anno and T. Logvinenko. Spherical DG-functors. *J. Eur. Math. Soc. (JEMS)*, 19(9):2577–2656, 2017.
- [AL25] R. Anno and T. Logvinenko. $\mathbb{P}^n$ -functors. *arXiv:1905.05740*, 2025.
- [Ann13] R. Anno. Spherical functors. *arXiv:0711.4409*, 2013.
- [Ara23] H. Arai. Half-spherical twists on derived categories of coherent sheaves. *arXiv:2302.12501*, 2023.
- [Bar20] F. Barbacovi. Spherical functors and the flop-flop autoequivalence. *arXiv:2007.14415*, 2020.
- [BB22] A. Bodzenta and A. Bondal. Flops and spherical functors. *Compos. Math.*, 158(5):1125–1187, 2022.
- [BB25] A. Bodzenta and A. Bondal. Categorifying non-commutative deformations. *J. Lond. Math. Soc. (2)*, 111(5):Paper No. e70176, 41, 2025.

- [BD24] A. Bodzenta and W. Donovan. Root stacks and periodic decompositions. *Manuscripta Math.*, 175(1-2):53–73, 2024.
- [BLS16] D. Bergh, V. A. Lunts, and O. M. Schnürer. Geometricity for derived categories of algebraic stacks. *Selecta Math. (N.S.)*, 22(4):2535–2568, 2016.
- [Bon89] A. I. Bondal. Representations of associative algebras and coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):25–44, 1989.
- [Bon18] A. Bondal. Schobers on orbifold Riemann surfaces. IHES Conference, Simons Collaboration on Homological Mirror Symmetry, 2018.
- [DKS23] T. Dyckerhoff, M. Kapranov, and V. Schechtman. N -spherical functors and categorification of Euler’s continuants. *arXiv:2306.13350 [math.CT]*, 2023.
- [Don13] W. Donovan. Grassmannian twists on the derived category via spherical functors. *Proc. Lond. Math. Soc. (3)*, 107(5):1053–1090, 2013.
- [Don24] W. Donovan. Derived symmetries for crepant contractions to hypersurfaces. *arXiv:2409.19555*, 2024.
- [DS14] W. Donovan and E. Segal. Window shifts, flop equivalences and Grassmannian twists. *Compos. Math.*, 150(6):942–978, 2014.
- [DW16] W. Donovan and M. Wemyss. Noncommutative deformations and flops. *Duke Math. J.*, 165(8):1397–1474, 2016.
- [DZ26] W. Donovan and L. Zheng. Quiver braid group action for a 3-fold crepant resolution. *arXiv:2512.19140*, 2026.
- [GW10] U. Görtz and T. Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.
- [HL15] D. Halpern-Leistner. The derived category of a GIT quotient. *J. Amer. Math. Soc.*, 28(3):871–912, 2015.
- [HLS16] D. Halpern-Leistner and I. Shipman. Autoequivalences of derived categories via geometric invariant theory. *Adv. Math.*, 303:1264–1299, 2016.
- [HT06] D. Huybrechts and R. Thomas. $\mathbb{P}$ -objects and autoequivalences of derived categories. *Math. Res. Lett.*, 13(1):87–98, 2006.
- [Huy06] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2006.
- [IU15] A. Ishii and K. Ueda. The special McKay correspondence and exceptional collections. *Tohoku Math. J. (2)*, 67(4):585–609, 2015.
- [Kel96] B. Keller. Derived categories and their uses. In *Handbook of algebra, Vol. 1*, volume 1 of *Handb. Algebr.*, pages 671–701. Elsevier/North-Holland, Amsterdam, 1996.
- [KL15] A. Kuznetsov and V. A. Lunts. Categorical resolutions of irrational singularities. *Int. Math. Res. Not. IMRN*, (13):4536–4625, 2015.
- [KS14] M. Kapranov and V. Schechtman. Perverse schobers. *arXiv:1411.2772v1 [math.AG]*, 2014.
- [Kuz19] A. Kuznetsov. Calabi-Yau and fractional Calabi-Yau categories. *J. Reine Angew. Math.*, 753:239–267, 2019.
- [Mea21] C. Meachan. A note on spherical functors. *Bull. Lond. Math. Soc.*, 53(3):956–962, 2021.
- [Plo05] D. Ploog. Autoequivalences of derived categories of smooth projective varieties. *PhD Thesis*, 2005.
- [Rei83] M. Reid. Minimal models of canonical 3-folds. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 131–180. North-Holland, Amsterdam, 1983.
- [Rou04] R. Rouquier. Categorification of the braid groups. *arXiv:0409593*, 2004.
- [Seg18] E. Segal. All autoequivalences are spherical twists. *Int. Math. Res. Not. IMRN*, (10):3137–3154, 2018.
- [ST01] P. Seidel and R. Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.
- [Tab05] G. Tabuada. Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. *C. R. Math. Acad. Sci. Paris*, 340(1):15–19, 2005.
- [Tod07] Y. Toda. On a certain generalization of spherical twists. *Bulletin de la Société Mathématique de France*, 135:119–134, 2007.

AGNIESZKA BODZENTA:

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, BANACHA 2, 02-097 WARSAW, POLAND.

Email address: A.Bodzenta@mimuw.edu.pl