Axiomatization of the Walk-Based Centrality Measures

An Extended Abstract of the PhD Dissertation

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Centrality measures constitute one of the fundamental tools of network analysis [19]. Their role is to assign to every node of a network a value that reflects the importance of this node. Centrality analysis finds numerous applications in the wide variety of fields: from social studies [13] and economics [2], through biology [18] and physics [31], to transportation [14] and computer science [21]. One of the most important centrality measures is PageRank [25], which was invented to determine the importance of webpages for Google web search engine, and thus had a great impact on the way the Internet looks.

However, what it means for a node to be important heavily depends on the context of a particular application. This, along with the ever growing number of proposed measures, makes the choice of a centrality to use a difficult task. Since different centrality measures return very different results, the problem of choosing an appropriate measure is of utmost importance. Hence, there is a need for research that will provide a better understanding of centrality measures and will help decide upon a centrality measure to use in a specific application.

A method that allows for achieving this goal is axiomatization. In this approach, we introduce a set of simple properties, called *axioms*, that characterize a given centrality measure. Then we formally prove that only this particular measure satisfies all of the axioms at the same time. In this way, we obtain an intuitive characterization of the centrality measure. Moreover, if all of axioms are desired in a particular application, then we know that this centrality measure is well-suited for this application (if any measure is). Consequently, if one of the axioms is invalid, another measure should be used instead.

In recent years, the axiomatic approach to centrality measures has been gaining popularity in the literature [4, 5, 6]. Axiomatic characterizations have been created for many centrality measures, such as closeness [30], beta measure [10], or attachment [29]. In particular, two axiomatizations were created for Seeley index, also called simplified PageRank [1, 26]. However, for many centrality measures such characterizations are still missing. Until recently, PageRank was also a measure without an axiomatization in the literature.

Against this background, the main contributions of this thesis are as follows: First, we introduce the first in the literature axiomatic characterization of PageRank. Next, we create a coherent axiomatization of three centrality measures: decay centrality, PageRank, and a novel measure—random walk decay centrality. Our analysis shows that while random walk decay centrality retains a majority of PageRank's properties, it may be more desirable than PageRank in some settings. Finally, we generalize our axiomatization of PageRank to create a consistent axiom system for four classic feedback centralities: eigenvector and Katz centralities, Seeley index, and PageRank.

1 Graphs and Centrality measures

Mathematical model of a network is a *graph*. Here, we consider mainly directed multigraphs with possible self-loops. Each such multigraph (later called simply a graph) is a pair, (V,E), where V is a finite and nonempty set of nodes, and E is a multiset of edges. For example, in the World Wide Web network, nodes represent pages and the edges hyperlinks between them (note that there can be multiple hyperlinks from one page to another). Each edge (u,v) is an *outgoing edge* of its *start*, u, and an *incoming edge* of its *end*, v.

Often, we are interested if it is possible to go from node u to node v through edges of a graph, i.e., if there exists a *path* from u to v. The length of a shortest such path is called a *distance* from u to v and it is denoted by $dist(u,v)$. If there is no path from u to v, we assume that $dist(u,v) = \infty$. Otherwise, node u is a *predecessor* of v and v a *successor* of u. Moreover, if $dist(u,v) = 1$, i.e., $(u,v) \in E$, then u is a *direct predecessor* of v and v a *direct successor* of u. We will denote the set of all direct predecessors of v by $P^1_v(G)$. If every two nodes in a graph are predecessors and successors of each other, then the graph is said to be *strongly connected*.

We also consider *node weights* that can be used to include additional information about nodes in a graph. In a context of World Wide Web network, node weights can model personal preferences of a user [25], how well a page fits into a given topic [16], or the fact that a page is trusted [15]. If no such information is available, one can assume uniform weights for all nodes. Formally, a graph with node weights is a pair (G,b) where $G = (V,E)$ is a graph and b is a function $b : V \to \mathbb{R}_{\geq 0}$ that assigns non-negative weight to each node.

A *centrality measure* is a function, F, that for a given graph, $G = (V,E)$, node weights b, and node $v \in V$, returns a real non-negative value $F_v(G,b)$ that represents the importance of node v in weighted graph (G,b).

1.1 Classic Centrality Measures

The three classic centrality measures are degree, closeness, and betweenness centralities [12, 24]. In this section, we state their definitions along with decay centrality which is a popular alternative for closeness centrality [19].

Most of the standard centrality measures were proposed for graphs without node weights. However, they can usually be easily adapted to this richer setting [22]. We will call versions of standard centrality measures for node-weighted graphs *personalized* [36].

Degree centrality [11] simply looks at the number of incoming edges to a node. In a personalized version instead of taking only the number of incoming edges, we sum the weights of their starts. Formally,

$$D_v(G,b) = \sum_{(u,v)\in E} b(u) = \sum_{u \in P^1_v(G)} b(u) \cdot \mu_G(u,v),$$

where $\mu_G(u,v)$ is a *multiplicity* of edge (u,v), i.e., the number of times edge (u,v) appears in multiset of edges E.

Closeness centrality [3] looks at the distances to a node from all the other nodes. It is defined as the reciprocal of the sum of these distances, hence nodes in the center of a graph have the highest values. In the personalized version, distance from each node is multiplied by its weight. In this way, the distances from nodes with large weights are more significant. Formally,

$$C_v(G,b) = \frac{1}{\sum_{u \in V \setminus \{v\}} b(u) \cdot dist(u,v)}.$$

Closeness centrality is usually defined for strongly connected graphs. Otherwise, if there exist nodes from which a node cannot be reached, then its centrality is always equal to zero.

Decay centrality [19] is a modification of closeness centrality that works for arbitrary graphs, not necessarily the strongly connected ones. Here, instead of looking at the sum of distances, each node at distance k contributes its weight multiplied by a^k, for some decay factor a. Formally, for $a \in [0,1)$, decay centrality is defined as follows:

$$Y_v(G,\beta) = \sum_{u \in V} b(u) \cdot a^{dist(u,v)}.$$

Finally, *betweenness centrality* [11] captures how often a node is an intermediary between two other nodes. To this end, for every pair of nodes, s,t, it counts what percentage of all shortest paths from s to t (we denote their number by σ_{st}), are shortest paths passing through a node in question, v (we denote their number by $\sigma_{st}(v)$). Then it multiplies each fraction by node weights of s and t and sums it for all $s,t \in V$. Formally,

$$B_v(G,\beta) = \sum_{s,t \in V : \sigma_{st} \neq 0} b(s)b(t) \cdot \frac{\sigma_{st}(v)}{\sigma_{st}}.$$

1.2 Feedback Centralities

Observe that all classic centrality measures are based on the shortest paths in a graph. This approach is adequate when the processes that occur in a network follow the shortest paths, which is the case in transportation or communication networks [9]. However, for many processes in the real-life networks, this assumption is clearly not adequate. A user surfing the Internet usually does not know how to reach another site with the minimal number of links [17]. Similarly, the news traveling through a social network moves in a complex, seemingly random way [23]. Hence, for these settings we need another type of centrality measures.

Feedback centralities form an appealing class of centrality measures that are not based on shortest paths. They assign centrality to a node recursively, based on its direct predecessors and their centrality.

The simplest feedback centrality is *eigenvector centrality* [8]. Here, the importance of each node is proportional to the total importance of its direct predecessors. Formally, eigenvector centrality is defined by a recursive equation:

$$EV_v(G,b) = \frac{1}{\lambda} \sum_{u \in P_v^1(G)} \mu_G(u,v) \cdot EV_u(G,b), \tag{1}$$

where λ is the largest eigenvalue of the adjacency matrix. For every strongly connected graph, Eq. (1) has a unique solution up to a scalar multiplication. Hence, some additional normalization condition is usually assumed to make the centrality measure well defined (e.g., that centralities of all nodes sum up to 1). In this work, we use a normalization that stems from the walk interpretation of eigenvector centrality, which is more consistent with other feedback centralities. Moreover, it allows us to define eigenvector centralities also on sums of disjoint strongly connected graphs with the same principal eigenvalues λ.

Another centrality based on a similar principle is *Katz centrality* [20]. Here, the importance of a node is mostly determined by the total importance of its direct predecessors, however an additional *basic importance* is added to every node, equal to its weight. Formally, for a decay factor $a \in \mathbb{R}_{\geq 0}$, Katz centrality is defined as a solution to:

$$K_v^a(G,b) = a \cdot \left(\sum_{u \in P_v^1(G)} \mu_G(u,v) \cdot K_u^a(G,b) \right) + b(v). \quad (2)$$

Adding a basic importance shifts the emphasis from the total importance of the direct predecessors back to their number. That is why Katz centrality is sometimes seen as a middle-ground between degree and eigenvector centralities. For a fixed a, Eq. (2) has the unique solution for all graphs with $\lambda < 1/a$.

In both eigenvector and Katz centralities, the whole importance of a node is "copied" to all of its direct successors. In turn, in *Seeley index* [28], which is also known as *Katz prestige* [19] or *simplified PageRank* [25], a node splits its importance equally among its successors. Consequently, the importance of predecessors is divided by their out-degree. Formally, Seeley index is defined as a solution to the following recursive equation:

$$SI_v(G,b) = \sum_{u \in P_v^1(G)} \frac{\mu_G(u,v)}{\deg_u^+(G)} \cdot SI_u(G,b). \quad (3)$$

Similarly to eigenvector centrality, Eq. (3) does not have a unique solution. Again, we normalize it using the walk interpretation. This allows us to uniquely define Seeley index for all sums of disjoint strongly connected graphs.

Finally, *PageRank* [25] modifies Seeley index by adding a basic importance to each node. In this way, for a decay factor $a \in [0,1)$, PageRank is uniquely defined for all graphs as follows:

$$PR_v^a(G,b) = a \cdot \left(\sum_{u \in P_v^1(G)} \frac{\mu_G(u,v)}{\deg_u^+(G)} \cdot PR_u^a(G,b) \right) + b(v).$$

Interestingly, each feedback centrality can be alternatively defined using *walks* on a graph, i.e., any sequence of nodes such that consecutive nodes are connected by an edge, including these with repeating nodes [33]. As a result, they are more suitable to model the importance of nodes in networks in which processes follow chaotic patterns, like the Internet traffic or news propagation in social media.

2 Axiomatization of PageRank

PageRank, originally proposed by Page et al. (1999) [25] as a measure of webpage importance, inspired researchers from various fields to use it in their own network problems. It was applied to indicate the most influential users on social media platforms [35], to assess the prestige of scientific journal in the citation network [7], to find the key proteins in metabolic networks [18], or even to determine the best tennis players in the history based on the network of their matches [27]. However, it is not clear if PageRank is a well-suited method to be used in these applications or whether other centrality measure would be more suitable.

With this problem in mind, we create the first axiomatic characterization of PageRank. Specifically, we introduce six intuitive axioms, which a centrality measure can satisfy. Furthermore, we prove that PageRank is the only centrality measure that satisfies all of them at once. In this way, we create new theoretical underpinnings for PageRank.

First five of our proposed axioms are *invariance* axioms, i.e., they consider a graph operation under which the centrality of certain nodes is invariant. The first two such axioms concern simple operations of removing a node and an edge from a graph:

> ***Node Deletion***: *removing an isolated node from a graph does not affect centralities of other nodes in the graph.*

> ***Edge Deletion***: *removing an edge from a graph does not affect centralities of nodes which are not successors of the start of this edge.*

In the setting of WWW network, Node Deletion, would mean that a webpage without any links and to which there are no links, e.g., a hidden resource on a server, does not have any impact on importance of other webpages. In turn, Edge Deletion states that removing a link from a webpage, A, can affect the importance of only these webpages that are reachable from A.

The next two axioms consider manipulations on edges in a graph:

> ***Edge Multiplication***: *creating additional copies of all of the outgoing edges of a node does not affect the centrality of any node in the graph.*

> ***Edge Swap***: *swapping the ends of two outgoing edges of nodes with equal centralities and out-degrees does not affect the centrality of any node in the graph.*

Edge Multiplication states that a centrality does not account for absolute number of outgoing edges. Indeed, if we change the number of outgoing edges of a node, without affecting the ratio of edges going to particular nodes, the centrality of every node in the graph remains the same. This is an important property in the WWW network, where the cost of creating a link is negligible. Edge Swap captures the intuition behind the feedback centralities: the gain in the centrality that a node receives from an incoming edge, depends only on the centrality and the out-degree of the start of this edge.

For our next axiom, Node Redirect, consider two webpages with identical contents and links. The axiom states that if we remove one such webpage and redirect the Internet traffic that goes through it to the other webpage, then their importance is summed up and the importance of other webpages is not affected. In terms of graphs, two nodes with the same outgoing edges are called *out-twins*. Now, *redirecting* a node, u, into its out-twin, v, means removing node u and transferring its weight and incoming edges to node v.

> ***Node Redirect***: *redirecting a node into its out-twin sums up their centralities and does not affect the centrality of other nodes in the graph.*

Our first five invariance axioms characterize PageRank up to a scalar multiplication. To make the characterization unique, we add our last normalization axiom, *Baseline*.

> ***Baseline***: *the centrality of an isolated node is equal to its weight.*

Our main result of this part of the thesis is as follows:

Theorem 1. *A centrality measure satisfies Node Deletion, Edge Deletion, Edge Multiplication, Edge Swap, Node Redirect, and Baseline if and only if it is PageRank.*

Our characterization can help decide whether PageRank should be used in a particular application. As an example, consider a network of tennis players analyzed by Radicchi [27]: for every tennis match in which A won with B, there is an edge from B to A. In such a network, PageRank indicates some node as the most central. Then, from Edge Multiplication we know that multiplying its outgoing edges 9000 times does not change the importance of any node in the network. However, this means that this player, who now lost most of his matches, is still considered the most central. Hence, a direct consequence of our characterization is the fact that PageRank is not suitable for such applications.

Axiom	PageRank	Random Walk Decay	Decay
Locality	+	+	+
Sink Merging	+	+	+
Directed Leaf Proportionality	+	+	+
One-Node Graph	+	+	+
Random Walk Property	+	+	-
Shortest Paths Property	-	-	+
Lack of Self-Impact	-	+	+
Edge Swap	+	-	-

Table 1: Our axiomatic characterizations of PageRank, random walk decay centrality, and decay centrality. The plus sign (+) indicates that the centrality measure satisfies the axiom, whereas the minus sign (-) indicates that the centrality measure violates it.

3 Random Walk Decay Centrality

PageRank is known for its random walk interpretation. Imagine a surfer that is browsing the Internet in a very schematic way: It starts from the random webpage and then, at each step of its walk, with probability a it randomly chooses one of the hyperlinks on a webpage it sees and follows it to the next webpage. At the same time, with probability $1-a$ it stops walking altogether. The expected number of times the surfer visits a webpage is equal to PageRank of this webpage (up to a scalar multiplication). This is undesirable in settings in which nodes decide upon their outgoing edges (e.g., users on a social media that decide whom to follow, or webpages that decide upon their links), as they can manipulate their PageRank by changing their outgoing edges.

Motivated by this observation, we introduce *random walk decay centrality*. Instead of the expected number of visits in a node it is equal to the probability that a node is visited at all (up to a scalar multiplication). This way, the outgoing edges of a node do not affect its centrality. Next, we create an axiomatic characterization of random walk decay centrality consistent with characterizations of PageRank and decay centrality. Our results are summarized in Table 1.

The axioms characterizing random walk decay centrality are as follows:

> ***Random Walk Property****: centrality depends solely on random walks on a graph.*
>
> ***Locality****: the centrality of a node depends only on the connected component to which this node belongs.*
>
> ***Sink Merging****: redirecting a sink into another sink without a common predecessor sums up their centralities and does not affect the centrality of other nodes.*
>
> ***Lack of Self-Impact****: the centrality of a node does not depend on its outgoing edges.*
>
> ***Directed Leaf Proportionality****: adding an edge from a sink to an isolated node increases the centrality of this node by the centrality of the sink times a constant* $a \in [0,1)$.
>
> ***One-Node Graph****: the centrality of a node with unit weight in a graph without any edges and other nodes is equal to one.*

Theorem 2. *A centrality measure satisfies Random Walk Property, Locality, Sink Merging, Lack of Self-Impact, Directed Leaf Proportionality, and One-Node Graph if and only if it is random walk decay centrality.*

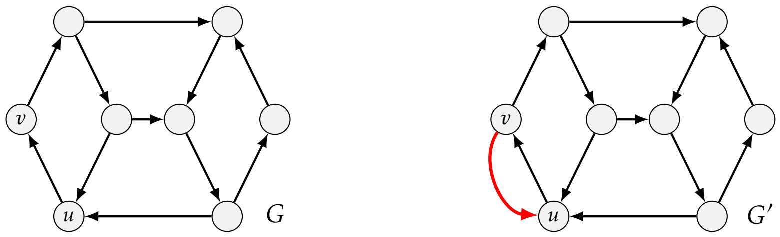

Figure 1: Example graphs illustrating the meaning of Lack of Self-Impact axiom. We assume uniform node weights in both $G = (V, E)$ and $G' = (V, E \cup \{(v, u)\})$, i.e., $b(w) = 1$, for every $w \in V$. In graph G, node v has the 5^{th} highest centrality according to both PageRank and random walk decay centrality ($PR^{0.9}_v(G, b) = 9.52$ and $RWD^{0.9}_v(G, b) = 5.13$). In graph G', edge (v, u) is added. Since random walk decay centrality satisfies Lack of Self-Impact, we know that such operation does not affect the centrality of v ($RWD^{0.9}_v(G', b) = 5.13$) and in fact it is still ranked as 5^{th} most central node. However, PageRank does not satisfy Lack of Self-Impact and v has much higher PageRank in G' than in G ($PR^{0.9}_v(G', b) = 14.08$). In fact, in G', PageRank ranks v as the second most central node.

If from axiomatic characterization of random walk decay centrality we remove Lack of Self-Impact and instead add Edge Swap, we obtain a unique characterization of PageRank.

Theorem 3. *A centrality measure satisfies Random Walk Property, Locality, Sink Merging, Edge Swap, Directed Leaf Proportionality, and One-Node Graph if and only if it is PageRank.*

Random walk decay centrality retains a majority of PageRank's properties. The only differences lie in Lack of Self-Impact and Edge Swap axioms. As we have discussed, the fact that random walk decay centrality satisfies Lack of Self-Impact, corresponds to its nonmanipulability (see Figure 1 for an illustrative example). On the other hand, the fact that random walk decay centrality does not satisfy Edge Swap allows it to take into account also other aspects of nodes, not only the centralities and out-degrees of its direct predecessors. As a result, random walk decay centrality is higher for nodes with direct predecessors belonging to a diverse set of communities in a network.

In a similar way, if in axiomatization of random walk decay centrality we exchange Random Walk Property for a new axiom, *Shortest Paths Property*, that states that the centrality is based on the shortest paths, then such set of axioms uniquely characterizes standard decay centrality. Lack of Self-Impact is implied by Shortest Paths Property, therefore it is redundant.

Theorem 4. *A centrality measure satisfies Shortest Paths Property, Locality, Sink Merging, Directed Leaf Proportionality, and One-Node Graph if and only if it is decay centrality.*

4 An Axiom System for Feedback Centralities

In the last part of the thesis, we extend our axiomatic characterization of PageRank in order to create a joint axiom system for all four main feedback centralities: eigenvector centrality, Katz centrality, Seeley index, and PageRank. To this end, we propose seven axioms: three general axioms satisfied by all four centralities, two one-node-modification axioms, and two borderline axioms. We prove that each of the four measures can be uniquely characterized by a subset of five axioms: three general ones, one

Centrality	General axioms	Node-modification axiom	Borderline axiom
Eigenvector	LOC, ED, NC	Edge Compensation	Cycle
Katz	LOC, ED, NC	Edge Compensation	Baseline
Seeley index	LOC, ED, NC	Edge Multiplication	Cycle
PageRank	LOC, ED, NC	Edge Multiplication	Baseline

Table 2: Our axiomatic characterizations of eigenvector centrality, Katz centrality, Seeley index, and PageRank. General axioms are Locality (LOC), Edge Deletion (ED), and Node Combination (NC).

one-node-modification axiom, and one borderline axiom. Our results are summarized in Table 2.

To study the axiomatization of the four centrality measures, we generalize the class of graphs we consider. Instead of multigraphs with node weights, we consider graphs with both node weights and edge weights. Formally, a *weighted graph* is a pair, (G,θ), where $G = (V,E)$ is a simple graph with set of nodes V and set of edges $E \subseteq V \times V$ and weights are a pair $\theta = (b,\mu)$, where $b : V \to \mathbb{R}_{\geq 0}$ are node weights and $\mu : E \to \mathbb{R}_{>0}$ are edge weights. Observe that multigraphs considered earlier can be seen as weighted graphs with natural numbers as edge weights only. Note that all centrality measure definitions still hold, but with μ instead of μ_G and the out-degree equal to the sum of weights of outgoing edges, i.e., $\deg^+_u(G) = \sum_{(u,v)\in E} \mu(u,v)$.

To characterize the centralities in question we consider a system of seven axioms. We note that among these four measures, only PageRank is well defined on all possible graphs.[1] Hence, in fact, we consider a weaker version of each axiom with an additional condition that for each graph considered in the axiom, the centrality is well defined.

Three of our axioms are general axioms satisfied by all four measures. The first two of them, are Edge Deletion and Locality.

For the next axiom, we consider a generalization of node redirection that works also on nodes that are not necessarily out-twins. To this end, we define *proportional combination* of two nodes, which preserves the significance of their outgoing edges. For centrality F, graph (G,θ), and two nodes u,w, the graph resulting from proportional combining of u into w is obtained in two steps: scaling weights of the outgoing edges of u and w proportionally to their centralities; and contracting node u into node w. What is important, if u and w are out-twins, then, for every centrality F, proportional combining of u into w reduces to redirecting u into w. Now, *Node Combination* states that if u, w, and all of their successors have equal out-degrees, then such an operation does not affect centralities in a graph.

> ***Node Combination****: proportional combining of nodes with equal out-degrees and equal out-degrees of successors sums up their centralities and does not affect the centralities of other nodes.*

Our next two axioms consider modification of the weights of edges in a graph. The first axiom is Edge Multiplication from adapted for graphs with edge weights. It states that multiplying the weights of the outgoing edges of a node by a constant does not affect the centrality of any node. It is satisfied by PageRank and Seeley index and it is not satisfied by Katz and eigenvector centralities. For them, we propose a similar axiom, Edge Compensation, emphasizing a different role of edge weights in determining centrality:

[1] Eigenvector centrality is defined on sums of disjoint strongly connected graphs with equal principal eigenvalues, Seeley index on sums of disjoint strongly connected graphs, and Katz centrality with decay factor $a \in \mathbb{R}_{\geq 0}$ on graphs with principal eigenvalue $\lambda < 1/a$.

***Edge Compensation**: multiplying the weights of the outgoing edges of a node by a constant and dividing its weight and the weights of its incoming edges by the same constant, divides the centrality of this node by the same constant and does not affect the centralities of the other nodes.*

Finally, our last two axioms address borderline cases. The first axiom is Baseline. It is satisfied by PageRank and Katz centrality. However, eigenvector centrality and Seeley index are both defined for sums of disjoint strongly connected graphs and there are no isolated nodes in such graphs. Thus, we propose a new axiom, Cycle, where we consider the simplest possible strongly connected graph—a cycle with uniform edge weights:

***Cycle**: the centrality of a node in a cycle graph with uniform edge weights is the arithmetic average of the weights of all the nodes in the graph.*

The following theorems indicate unique characterizations of each of the centrality measures in question based on our axiom system.

Theorem 5. *A centrality measure defined on sums of disjoint strongly connected graphs satisfies Locality, Edge Deletion, Node Combination, Edge Multiplication, and Cycle if and only if it is Seeley index.*

Theorem 6. *A centrality measure defined on sums of disjoint strongly connected graphs with equal principal eigenvalues satisfies Locality, Edge Deletion, Node Combination, Edge Compensation, and Cycle if and only if it is eigenvector centrality.*

Theorem 7. *A centrality measure satisfies Locality, Edge Deletion, Node Combination, Edge Multiplication, and Baseline if and only if it is PageRank.*

Theorem 8. *For every $a \in \mathbb{R}_{\geq 0}$, a centrality measure defined on graphs with principal eigenvalue $\lambda < 1/a$ satisfies Locality, Edge Deletion, Node Combination, Edge Compensation, and Baseline if and only if it is Katz centrality.*

5 Papers Composing This Thesis

The thesis is based on the following papers:

- Section 2: *Axiomatization of the PageRank Centrality*, paper co-authored with Oskar Skibski, published in the proceedings of the **IJCAI-18** conference [34].
- Section 3: *Random Walk Decay Centrality*, paper co-authored with Talal Rahwan and Oskar Skibski, published in the proceedings of the **AAAI-19** conference [32].
- Section 4: *An Axiom System for Feedback Centralities*, paper co-authored with Oskar Skibski, to be published in the proceedings of the **IJCAI-21** conference [33].

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