PESEL: | | | | | | | | | | | |

University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Exam for 2nd cycle studies of MACHINE LEARNING

September 20, 2022

Solving time: 150 minutes

In each of the 30 problems there are three variants: (a), (b), and (c). For each variant you should answer if it is true, writing **YES** *or* **NO** *in the box to the left of it. In the case of an error you should cross out the box and write the correct word on its left side.*

Example of a correctly solved problem

4. Every integer of the form $10^n - 1$, where n is integer and positive,

   $\boxed{\texttt{YES}}$ (a) is divisible by 9;

   $\boxed{\texttt{NO}}$ (b) is prime;

   $\boxed{\texttt{YES}}$ (c) is odd.

You can write only in the indicated places and only the words **YES** *and* **NO**. *Use pen.*

Scoring

You get "big" points (0 – 30) and "small" points (0 – 90):

- *one "big" point for each problem, in which you correctly solved all three variants;*
- *one "small" point for each correctly solved variant. So 3 "small" points in a single problem give one "big" point.*

The final result of the exam is the number

$$W = \min(30, D + m/100),$$

where D is the number of "big" points, and m is the number of "small" points, e.g. score 5.50 means that a candidate correctly solved 50 variants in the whole test, but gave correct answers to all three variants in a set for some five problems.

"Big" points are more important. "Small" points are just to increase resolution in the case when many candidates get the same number of "big" points.

Good luck!

1. Function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(0) = 0$. It follows that

   - [ ] (a) f has a local extremum at $x = 0$;
   - [ ] (b) if $f'(x) < 0$ for all $x \neq 0$ then f is decreasing in $\mathbb{R}$;
   - [ ] (c) f is continuous at $x = 2022$.

2. Let $\langle S_n \rangle_{n \geq 1}$ be a sequence of partial sums of a divergent series. It follows that the sequence $\langle S_{n+1} - S_n \rangle_{n \geq 1}$ is

   - [ ] (a) divergent;
   - [ ] (b) unbounded;
   - [ ] (c) not convergent to 0.

3. Let X be a linear space of dimension 15 over the field of complex numbers. It follows that

   - [ ] (a) every set of 20 vectors from X is linearly dependent;
   - [ ] (b) every set of 10 vectors from X is linearly independent;
   - [ ] (c) every base of X consists of 15 vectors.

4. Consider space $\mathcal{P}$ of real polynomials of degree at most 1. Define functionals φ_1 and φ_2 as $\varphi_1(p) = p(0)$, $\varphi_2(p) = p(1)$ for $p \in \mathcal{P}$. It follows that

   - [ ] (a) functionals φ_1 and φ_2 are linearly independent;
   - [ ] (b) functional φ given by $\varphi(p) = p'(\frac{1}{2})$ is a linear combination of φ_1 and φ_2;
   - [ ] (c) polynomials $p_1(t) = 1 - t$, $p_2(t) = t$ and functionals φ_1, φ_2 form dual bases in $\mathcal{P}$ and the dual space $\mathcal{P}^*$, respectively.

5. The system of linear equations $Ax = b$ with matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -5 & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

   - [ ] (a) has exactly one solution for $b = [1, 7, 4]^T$;
   - [ ] (b) has the same number of solutions for each vector $b \in \mathbb{R}^3$;
   - [ ] (c) has a nonempty and bounded set of solutions for some vector $b \in \mathbb{R}^3$.

6. Let X be an Euclidean space and vectors $u, v, w \in X$ form an orthonormal set. It follows that

   - [ ] (a) vectors u, v, w are linearly independent;
   - [ ] (b) vectors $u + v + w$ and $u - 2v + w$ are orthogonal;
   - [ ] (c) vectors $u + v + w$ and $u - 2v + w$ are orthonormal.

7. Let $f\colon \mathbb{N} \to \mathbb{N}$ and let f^{2022} be the 2022-fold composition of f. It follows that

   - [ ] (a) f is an injection iff f^{2022} is an injection;
   - [ ] (b) f is a surjection iff f^{2022} is a surjection;
   - [ ] (c) $f(42) = 42$ iff $f^{2022}(42) = 42$.

8. Let A be a nonempty set and let $r \subseteq A \times A$ be an acyclic relation. It follows that

   - [ ] (a) r can be extended to a partial order on A;
   - [ ] (b) r can be extended to a well founded order on A;
   - [ ] (c) r can be extended to a linear order on A.

9. Let n be a positive integer. The number of placements of n indistinguishable balls in three distinguishable bins, so that

   - [ ] (a) each bin is nonempty, equals $\binom{n-1}{2}$;
   - [ ] (b) the first bin contains more balls than the last bin, equals $\frac{1}{2}\binom{n-1}{2}$;
   - [ ] (c) each bin contains an even number of balls, equals $\frac{(-1)^n+1}{2}\binom{n/2+2}{2}$.

10. Let $A(x)$ be the (ordinary) generating function of sequence $\langle a_n \rangle_{n=0}^{\infty}$. It follows that the generating function of the sequence

    - [ ] (a) $\langle -a_n \rangle_{n=0}^{\infty}$ is $A(-x)$;
    - [ ] (b) $\langle 2^n a_n \rangle_{n=0}^{\infty}$ is $A(2x)$;
    - [ ] (c) $\langle \sum_{k=0}^{n} a_k \rangle_{n=0}^{\infty}$ is $\frac{A(x)}{1-x}$.

11. Let $\overline{G}$ denote the complement of graph G, and let $\chi(G)$ denote its chromatic number. Let $K_{m,n}$ be the complete bipartite graph with parts of size m and n, for $m, n \geq 1$. It follows that

    - [ ] (a) $\chi(K_{m,n}) \geq \max(m, n)$;
    - [ ] (b) $\chi(\overline{K_{m,n}}) = \max(m, n)$;
    - [ ] (c) if $m + n > 2022$ then $K_{m,n}$ is nonplanar.

12. For integer $n > 1$ and integers a and b consider system of congruences $x \equiv a \pmod{n}$, $x \equiv b \pmod{n^2}$. This system

    - [ ] (a) has a solution if $a = b$;
    - [ ] (b) has no solutions if $a \neq b$;
    - [ ] (c) has at most one solution in the set $\{0, 1, 2, \ldots, n^2 - 1\}$.

13. We randomly draw 3 balls from a set of 3 white balls and 4 black balls. The probability of drawing at least 2 white balls

    - [ ] (a) is greater for drawing with replacement than for drawing without replacement;
    - [ ] (b) is less than $\frac{1}{2}$, both with and without replacement;
    - [ ] (c) with replacement is less than the probability of drawing exactly 2 black balls with replacement.

14. Let X, Y be independent random variables with positive integer values and let $g_X(t)$, $g_Y(t)$ be their probability generating functions. It follows that

- [ ] (a) $g_X(t) + g_Y(t)$ is the probability generating function of the variable $X + Y$;
- [ ] (b) $g_X(t) \cdot g_Y(t)$ is the probability generating function of the variable $X \cdot Y$;
- [ ] (c) $g_X(g_Y(t))$ is the probability generating function of the variable Y^X.

15. There exists a probabilistic space Ω and events $A, B \subseteq \Omega$ such that $P(A) = P(B) = \frac{2}{3}$ and

- [ ] (a) A and B are independent;
- [ ] (b) $P(A|B) = \frac{1}{3}$;
- [ ] (c) $P(A|B) \neq P(B|A)$;

16. Let X, Y be independent random variables. It follows that

- [ ] (a) if both X and Y have Poisson distribution then so has $X - Y$;
- [ ] (b) if both X and Y have geometric distribution then so has $X - Y$;
- [ ] (c) if both X and Y have normal distribution then so has $X - Y$.

17. Let X and Y be random variables such that $E(X)$ and $E(Y)$ are finite and $E(X \cdot Y) = E(X) \cdot E(Y)$. It follows that

- [ ] (a) X and Y are independent;
- [ ] (b) $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$;
- [ ] (c) $E(X + Y) = E(X) + E(Y)$.

18. Given $a \in \mathbb{R}$, we search for real roots of a nonlinear equation $x^3 - x = a$ by Newton's method.

- [ ] (a) The formula for the next iteration of the method is $x_{k+1} = \frac{2x_k^3 - a}{3x_k^2 + 1}$.
- [ ] (b) For $a = 1$ and any $x_0 \geq 1$, Newton's method is convergent to some root of the equation.
- [ ] (c) For $a = 0$ and $x_0 = 10^{-2}$ Newton's method is quadratically convergent to zero.

19. The following function is interpolated by polynomial $w(x) = 1 - x^2 + x(x-1)(x+1)^9$ at points -1, 0, 1:

- [ ] (a) $\sin(\pi x)$;
- [ ] (b) $\cos(\frac{1}{2}\pi x)$;
- [ ] (c) $\cos(2\pi x)$.

20. The Internet Protocol (IP) uses the mechanism of

- [ ] (a) fragmentation;
- [ ] (b) retransmission;
- [ ] (c) flow control.

21. Recall that the *height* of a rooted tree is the number of *edges* on the longest path from the root to a leaf. G is a 2-vertex-connected graph on 1001 vertices. It follows that

- (a) if the height of some DFS tree of G is 1000 then G contains a Hamilton cycle;
- (b) the height of every BFS tree of G does not exceed 500;
- (c) the height of every DFS tree of G is different from 2.

22. Let T be the Fibonacci heap of type MIN obtained by inserting in turn numbers 1, 2, $\ldots$, 2022 to the initially empty heap, and let T' be obtained from T by applying operation DeleteMin. It follows that

- (a) T consists of at most 8 trees;
- (b) T' consists of at most 8 trees;
- (c) the height of the highest tree in T' is at least 10.

23. Let `R` be a table with column `A` and r rows, and let `S` be a table with column `A` and s rows. It follows that query `SELECT * FROM R, S WHERE R.A = S.A` returns

- (a) at least $\min(r, s)$ rows;
- (b) at most $\max(r, s)$ rows;
- (c) exactly $r \cdot s$ rows.

24. Suppose that query `SELECT R.A FROM R` returns r rows, query `SELECT S.A FROM S` returns s rows, and query `SELECT R.A FROM R WHERE R.A NOT IN (SELECT S.A FROM S)` returns k rows. It follows that

- (a) $k = r - s$;
- (b) $r - s \leq k \leq r$;
- (c) $0 \leq k \leq r$.

25. Consider the following program in Python:

```
1  class Fruit:
2      def __init__(self, species):
3          self.species = species
4
5  class Apple(Fruit):
6      def __init__(self, species, colour):
7          self.colour = colour
8
9  #frt = Fruit("Reneta");          print(getattr(frt, species))
10 #frt = Fruit("Reneta", "red");   print(getattr(frt, "species"))
11 #frt = Apple("Reneta", "red");   print(getattr(frt, "colour"))
```

After uncommenting

- (a) line 9 the program when executed prints the string `Reneta`;
- (b) line 10 the program when executed prints the string `Reneta`;
- (c) line 11 the program when executed prints the string `red`.

26. Consider the following program in Python:

```
1  k = 0
2
3  def f1():
4      return k + 1
5
6  def f2():
7      k = 1
8
9  def f3(k):
10     k = 1
11
12 #k = f1()
13 #f2()
14 #f3(k)
15
16 print(k)
```

After uncommenting

(a) line 12 the program when executed prints `1`;

(b) line 13 the program when executed prints `1`;

(c) line 14 the program when executed prints `1`.

27. Consider the following program in Python:

```
1  def f1():
2      lst = []
3      for i in range(5):
4          lst += [i]
5      print(5 in lst)
6
7  def f2():
8      lst = []
9      lst.append([1])
10     print(1 in lst)
11
12 def f3():
13     lst = [1]
14     lst *= 2
15     print(2 in lst)
16
17 #f1()
18 #f2()
19 #f3()
```

After uncommenting

(a) line 17 the program when executed prints `True`;

(b) line 18 the program when executed prints `True`;

(c) line 19 the program when executed prints `True`.

28. Consider a computer system with N processes using N nonshareable resources, where $N > 1$. Process $W(k)$ requires k resources, where $1 \leq k \leq N$. Let S be a strongly fair semaphore. Consider the following algorithm written in pseudocode:

```
process W(k:1..N)
begin
  while true begin
    local_section;
    for i := 1 to k do P(S);
    using_resources;
    for i := 1 to k do V(S);
  end;
end;
```

(a) If S is initialized with value N then this algorithm has safety property.

(b) If S is initialized with value N then this algorithm has liveness property.

(c) If S is initialized with value $N + 1$ then this algorithm has safety property.

29. In some processor, the virtual addresses are 48-bit long. The page size, the page table size, and the page table directory size are 256 KiB. The size of a single entry in the page table and in the page table directory is 8 B. The virtual address is divided into three fields, where

(a) the offset has 17 bits;

(b) the page number has 16 bits;

(c) the page table number in the page table directory has 15 bits.

30. For a deadlock to occur it is sufficient that

(a) at least one resource is nonshareable;

(b) a process holding a resource cannot be forced to release it;

(c) a process is holding at least one resource while waiting for another resources.