Between the Pytkeev and Fréchet-Urysohn properties of function spaces

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Based on a joint work in progress with S. Bardyla and J. Šupina.
Fréchet-Urysohn spaces

All spaces are Tychonoff.

Definition

- $Y$ is a Fréchet-Urysohn space (or, equivalently, has the Fréchet-Urysohn property) if for every $A \subset Y$ and $y \in \overline{A} \setminus A$ there exists a sequence $\langle a_n : n \in \omega \rangle \in A^\omega$ converging to $y$.

- $Y$ has countable tightness (equivalently, $t(Y) \leq \omega$) if for every $A \subset Y$ and $y \in \overline{A} \setminus A$ there exists $B \in [A]^\omega$ with $y \in \overline{B}$.

$Y$ is FU $\implies$ $t(Y) \leq \omega$

Example. All metrizable spaces are FU. More generally, all first-countable spaces are FU. All spaces $X$ with $t(X) \leq \omega$ and character $\prec p$ are FU.

Theorem (Hrusak-Ramos Garcia 2014)

It is consistent that all separable FU topological groups are metrizable.
**Definition**

$C_p(X) = \{ f : X \to \mathbb{R} \mid f \text{ is continuous} \}$. We consider $C_p(X)$ with the topology inherited from the Tychonoff product $\mathbb{R}^X$.

**Easy Fact.** $C_p(X)$ is metrizable iff $C_p(X)$ is first-countable iff $|X| \leq \omega$.

**Theorem (Arkhangelskii-Pytkeev 1980s)**

$t(C_p(X)) \leq \omega$ iff $X^n$ is Lindelöf for all $n \in \omega$. In particular, $t(C_p(X)) \leq \omega$ for every metrizable separable space.
FU spaces of functions

Definition
A family $\mathcal{U} \subset \mathcal{P}(X)$ is called

- an $\omega$-cover of $X$, if $X \not\in \mathcal{U}$ and for every $K \in [X]^{<\omega}$ there exists $U \in \mathcal{U}$ such that $K \subset U$.

- a $\gamma$-cover of $X$, if for every $x \in X$, the set $\{U \in \mathcal{U} : x \not\in U\}$ is at most finite.

A space $X$ is called a $\gamma$-space, if every open $\omega$-cover of $X$ contains a $\gamma$-subcover.

Example. $\{(n, n + 2) : n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$ which is not an $\omega$-cover.
$\{(-n, n) : n \in \omega\}$ is a $\gamma$-cover of $\mathbb{R}$.
$\{U \subset \mathbb{R} : U \text{ is open and } \mu(U) < 1\}$ is an $\omega$-cover of $\mathbb{R}$ without any $\gamma$-subcover, so $\mathbb{R}$ is not a $\gamma$-space.

Theorem (Gerlits-Nagy 1982)
$C_p(X)$ is FU iff $X$ is a $\gamma$-space.
Intuition behind the proof of the Gerlits-Nagy theorem

For $U \subset X$ and $x \in X$ set $\chi_U(x) = 0$ if $x \in U$ and $\chi_U(x) = 1$ otherwise. Let $\mathcal{U}$ be a family of clopen subsets of $X$.

Easy to check:

$0 \in \{\chi_U : U \in \mathcal{U}\} \setminus \{\chi_U : U \in \mathcal{U}\}$ iff $\mathcal{U}$ is an $\omega$-cover of $X$.

Indeed, pick a basic open neighbourhood

$W := [1/2; x_0, \ldots, x_n] = \{f \in C_p(X) : \forall i \leq n(|f(x_i)| < 1/2)\}$ of

$0$ in $C_p(X)$ and note that $\chi_U \in W$ iff $\{x_0, \ldots, x_n\} \subset U$.

Equally easy to check:

For $\{U_n : n \in \omega\} \subset \mathcal{U}$, $\{\chi_{U_n} : n \in \omega\}$ converges to $0$ iff

$\{U_n : n \in \omega\}$ is a $\gamma$-cover of $X$.

This proves the Gerlits-Nagy theorem for $C_p(X, 2)$. 
Examples of $\gamma$-spaces

Observation
If $X$ is a metrizable separable space of size $< \mathfrak{p}$, then $X$ is a $\gamma$-space.

Theorem (Gerlits-Nagy 1982)
If $X \subset \mathbb{R}$ is a $\gamma$-space, then $X$ has the strong measure zero. In particular, there are no uncountable metrizable $\gamma$-spaces in the Laver model.

Theorem (Galvin-A. Miller 1984)
If $\mathfrak{p} = \mathfrak{c}$, then there exists a $\gamma$-space $X \subset 2^\omega$ of size $\mathfrak{p}$.

Theorem (Orenshtein-Tsaban 2011)
If $\mathfrak{p} = \mathfrak{b}$, then there exists a $\gamma$-space $X \subset 2^\omega$ of size $\mathfrak{p}$.

Previously was unknown even under $\mathfrak{d} = \omega_1$!
Theorem (A. Miller 2005)

In the Hechler model there are no uncountable metrizable \( \gamma \)-spaces. In particular, the existence of uncountable strong measure zero sets of reals does not imply the existence of uncountable \( \gamma \)-spaces of reals.

Theorem (A. Miller-Tsaban-Z. 2016)

Metrizable \( \gamma \)-spaces are preserved by Cohen forcing.

Later, in a joint work with Repovš we have introduced a property of proper posets which is satisfied by Cohen, Miller, and Sacks forcings, and such that metrizable \( \gamma \)-spaces are preserved by countable support iterations of posets with this property.

Question (A. Miller)

Suppose that \( X \subseteq \mathbb{R} \) is a \( \gamma \)-space and \( Y \subseteq \mathbb{R} \), \( |Y| < \mathfrak{p} \). Is \( X \times Y \) a \( \gamma \)-space? What if \( |Y| = \omega_1 \) and MA plus non-CH hold?
Weakening the FU property

Definition
A space $Y$ is

- **sequential**, if for every non-closed $A \subset Y$ there exists $y \in \overline{A} \setminus A$ and a sequence of elements of $A$ convergent to $y$.
- **subsequential**, if it can be embedded into a sequential space.
- **Pytkeev**, if for every $A \subset Y$ and $y \in \overline{A} \setminus A$ there exists a countable family $B$ of infinite subsets of $A$ such that for every open $O \ni y$ there exists $B \in B$ with $B \subset O$.

Proposition (Pytkeev 1984)
*Subsequential spaces have the Pytkeev property.*

Theorem (Pytkeev 1982)
$C_p(X)$ is sequential iff it is FU.
Weakening the FU property, continued

FU $\implies$ sequential $\implies$ subsequential $\implies$ Pykeev $\implies$ countable tightness.

Question (Arkhangelskii 198?)

*Is the subsequentiality equivalent to the FU property for $C_p$-spaces?*

Theorem (Malykhin 1999)

$C_p([0, 1])$ is not Pykeev, and hence it is not subsequential.

Theorem (A. Miller 2008)

*Let $X$ be a metrizable space. If $C_p(X)$ is Pykeev, then $X$ has the strong measure zero with respect to any totally bounded continuous metric on it. Therefore $X$ is zero-dimensional. Moreover, in the Laver model $|X| = \omega$ iff $C_p(X)$ is Pykeev.*

Question (Malykhin-Tironi 2000)

*Does there exist a non-subsequential Pykeev space in ZFC?*
M. Sakai 2006: Are the Pytkkeev and FU properties equivalent for $C_p$-spaces?

Theorem (Bardyla-Šupina-Z. 2020)

(CH) There exists $X \subset 2^\omega$ such that $C_p(X)$ has the Pytkkeev property but fails to be FU.

Theorem (Simon-Tsaban 2008)

The minimal cardinality of a set $X \subset \mathbb{R}$ such that $C_p(X)$ does not have the Pytkkeev property is equal to $\mathfrak{p}$.

Thus no solution to Sakai’s problem by playing around with cardinal characteristics.
Theorem (Simon-Tsaban 2008)

TFAE for a zero-dimensional space $X$:

- $C_p(X)$ has the Pytkeev property;
- Each clopen $\omega$-cover $\mathcal{U}$ of $X$ contains infinite subsets $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$ such that $\{\bigcap \mathcal{U}_n : n \in \omega\}$ is an $\omega$-cover of $X$. □

The latter property of $X$ will be denoted by $(\pi)$.

Note that if there exists an infinite $\mathcal{U}_\infty \subset \mathcal{U}$ such that $\mathcal{U}_\infty \subset^* \mathcal{U}_n$ for all $n$, then $\mathcal{U}_\infty$ is a $\gamma$-cover of $X$. This will give us later a hint which witnesses for $(\pi)$ we should have in a potential counterexample.

Remark. In the characterization above it is enough to demand that for all finite $K \subset X$ there exists $n$ such that $K \subset U$ for almost all $U \in \mathcal{U}_n$. 
Lemma (Galvin-Miller 1984)

Let $\mathcal{U}$ be an $\omega$-cover of $[\omega]^{<\omega}$ consisting of clopen subsets of $\mathcal{P}(\omega)$. Then there exist an increasing number sequence $\langle k_n : n \in \omega \rangle$ and $\langle U_n : n \in \omega \rangle \in \mathcal{U}^\omega$ such that for every $n$ and $x \subseteq \omega$, if $x \cap [k_n, k_{n+1}) = \emptyset$, then $x \in U_n$.

Proof. Enough to prove the following:

For every $k \in \omega$ there exists $k' > k$ and $U \in \mathcal{U}$ such that for every $x \subseteq \omega$, if $x \cap [k, k') = \emptyset$, then $x \in U$.

Hint: take $U \in \mathcal{U}$ such that $U \supseteq \mathcal{P}(k)$ and using that it is open find suitable $k'$.

$\square$.
Lemma

Let $\mathcal{G}$ be an ultrafilter on $\omega$. If the set $\{a_\alpha : \alpha < c\} \subset \mathcal{G}^*$ satisfies

$$ \forall x \in [\omega]^{\omega} \exists \alpha \ (|x \cap a_\alpha| = \omega), $$

then $X := \{a_\alpha : \alpha < c\} \cup [\omega]^{<\omega}$ is not a $\gamma$-space.

Proof. Note that $\mathcal{O} = \{O_n : n \in \omega\}$, where $O_n = \{x \subset \omega : n \not\in x\}$ is an $\omega$-cover of $X$. The assumption on $X$ states literally that $\mathcal{O}$ has no $\gamma$-subcover of $X$. $\Box$
Lemma

(CH). Let \( \{ k_\alpha : \alpha < \omega_1 \} \) be an enumeration of all increasing sequences from \( \omega \omega \) (each sequence repeated \( \omega_1 \) many times) and \( \mathcal{G} \) be an ultrafilter on \( \omega \). If for a partition \( \{ I_n : n \in \omega \} \subset [\omega]^\omega \) of \( \omega \), the set \( \{ a_\alpha : \alpha < \omega_1 \} \) satisfies

\[
\{ n \in \omega : (\forall^{\infty} j \in I_n) [k_\beta(j), k_\beta(j+1)) \cap a_\alpha = \emptyset \} \in \mathcal{G}
\]

for all \( \beta \leq \alpha \), then \( X := \{ a_\alpha : \alpha < \omega_1 \} \cup [\omega]^{<\omega} \) has \( (\pi) \).

Proof. Given an \( \omega \)-cover \( \mathcal{U} \) of \( X \), find \( \beta \) and \( \langle U_j : j \in \omega \rangle \in \mathcal{U}^{\omega} \) such that \( x \cap [k_\alpha(j), k_\alpha(j+1)) = \emptyset \) implies \( x \in U_j \). Set \( \mathcal{U}_n = \{ U_j : j \in I_n \} \) and pick a finite \( s \subset \omega_1 \setminus \beta \). For every \( \alpha \in s \) the set

\[
G_\alpha := \{ n \in \omega : (\forall^{\infty} j \in I_n) [k_\beta(j), k_\beta(j+1)) \cap a_\alpha = \emptyset \}
\]

belongs to \( \mathcal{G} \), and note that

\[
G_\alpha \subset \{ n \in \omega : (\forall^{\infty} j \in I_n) a_\alpha \in U_j \}.
\]

Pick \( n \in \bigcap_{\alpha \in s} G_\alpha \) and note that \( \{ a_\alpha : \alpha \in s \} \subset U_j \) for almost all \( j \in I_n \), i.e., for almost all \( U_j \in \mathcal{U}_n \). Thus,

\[
\{ \cap \mathcal{V} : \mathcal{V} \text{ is a cofinite subset of } \mathcal{U}_n \text{ for some } n \}
\]

is an \( \omega \)-cover of \( \{ a_\alpha : \alpha < \omega_1 \} \cup [\omega]^{<\omega} \). This is almost \( (\pi) \) for \( X : -) \) \( \square \)
Bringing together two halves

Let \( \{ k_\alpha : \alpha < \omega_1 \} \) be an enumeration of all increasing sequences from \( \omega \omega \) (each sequence repeated \( \omega_1 \) many times) and \( \mathcal{G} \) be an ultrafilter on \( \omega \). Let \( \{ I_n : n \in \omega \} \subset [\omega]^{\omega} \) be a partition of \( \omega \). If the set \( \{ a_\alpha : \alpha < \omega_1 \} \subset \mathcal{G}^* \) satisfies

\[
\{ n \in \omega : (\forall \infty j \in I_n) [k_\beta(j), k_\beta(j + 1)) \cap a_\alpha = \emptyset \} \in \mathcal{G}
\]

for all \( \beta \leq \alpha \), and

\[
\forall x \in [\omega]^\omega \exists \alpha (|x \cap a_\alpha| = \omega),
\]

then \( X := \{ a_\alpha : \alpha < c \} \cup [\omega]^{<\omega} \) is not a \( \gamma \)-space but still satisfies (\( \pi \)).

Now just do a rather straightforward transfinite construction to get the two conditions above fulfilled...

**Remark.** The construction could be done under \( p = c \), but there is a model of \( p = b \) where all sets we can get in **such** constructions are \( \gamma \)-sets.
Is our result optimal/satisfactory?

No!!!

The property \( \pi \) is typical combinatorial covering property (selection principle), and being a \( \gamma \)-space is the strongest “standard” one. The weakest “standard” selection principle is the following Menger property:

*For every sequence \( \langle U_n : n \in \omega \rangle \) of open covers of \( X \) there is a sequence \( \langle V_n : n \in \omega \rangle \) such that \( V_n \in [U_n]^{<\omega} \) and \( \{ \cup V_n : n \in \omega \} \) is a cover of \( X \).*

**Question**

Does \( \pi \) imply Menger? What about the subsets of \( 2^\omega \)?

Would be very helpful (among other, sufficient for the answer of the question above) to know the answer to the following

**Question**

Is \( \pi \) for subsets of \( 2^\omega \) preserved by the Cohen forcing?

If yes, then in ZFC we would be able to conclude that \( \pi \) implies all finite powers being both Rothberger and Hurewicz, thus pushing it rather close to \( \gamma \)-spaces in the Scheepers diagram.
Dziękuuję za uwagę.

Thank you for your attention.