Sharp inequalities for the Ornstein-Uhlenbeck operator

Vít Musil (jointly with A. Cianchi and L. Pick)

The Ornstein-Uhlenbeck operator $\mathcal{L} = \Delta - x \cdot \nabla$ is the natural counterpart of the Laplace operator when the ambient Euclidean space is replaced by the probability space $(\mathbb{R}^n, \gamma_n)$, where $\gamma_n$ denotes the Gauss measure with the density

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} \, dx.$$ 

For any $f \in L^1(\mathbb{R}^n, \gamma_n)$ satisfying $\int_{\mathbb{R}^n} f \, d\gamma_n = 0$ a unique solution to

$$\mathcal{L}u = -f \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \text{med}(u) = 0$$

exists (in a suitable weak sense) and the optimal transfer of integrability from $f$ to $u$ is available. More precisely, for a given r.i. space $X$, we characterise the optimal (smallest) r.i. space $Y$ such that

$$\|u\|_{Y(\mathbb{R}^n, \gamma_n)} \leq C\|f\|_{X(\mathbb{R}^n, \gamma_n)}$$

for some $C > 0$ and every $f \in X(\mathbb{R}^n, \gamma_n)$. Unlike in the Euclidean case, the gain of integrability is not always guaranteed. For instance, if $f$ belongs to the exponential space $\exp L^\beta$, $\beta > 0$, the Orlicz space built upon a Young function equivalent to $e^{\beta}$ near infinity, the increase of integrability of $u$ deteriorates so that only $u \in \exp L^\beta$, i.e.

$$\|u\|_{\exp L^\beta(\mathbb{R}^n, \gamma_n)} \leq C\|f\|_{\exp L^\beta(\mathbb{R}^n, \gamma_n)}.$$

Our specific concern is the sharp form of the last inequality. We identify the largest constant $\theta$ in the integral inequality

$$\sup_u \int_{\mathbb{R}^n} \exp^\beta(\theta|u|) \, d\gamma_n < \infty,$$

where the supremum is extended over all $u$ subject to a constraint

$$\int_{\mathbb{R}^n} \exp^\beta(|\mathcal{L}u|) \, d\gamma_n \leq M \quad \text{and} \quad \text{med}(u) = 0$$

for some $M > 1$. We also show that the maximizers exist in relevant cases, i.e. that the supremum is in fact attained.

This problem can be regarded as a Gaussian analogue of that solved by Adams for the classical Laplacian in the Euclidean setting, which is in turn a second order version of the famous Moser’s inequality.