"The tragedy of the commons" in the dynamic context

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Games
Nash eq.
Stackelberg eq.
Pareto opt.
Example-static
The tragedy

Dynamic games
A simple model
Objectives

Bellman
Revised sufficient condition

Social optimum
Nash equilibrium
Nash equilibria for \( n \) players
Finite horizon truncation

Enforcing optimality
Carrying capacity

Numerics

Continuous time
Conclusions
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Extraction of a common fishery
The motivating example

Extraction of a common fishery with possibility of extinction, division into Exclusive Economic Zones
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Extraction of a common fishery with possibility of extinction, division into Exclusive Economic Zones and inherent constraints
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Extraction of a common fishery with possibility of extinction, division into Exclusive Economic Zones and inherent constraints with possibility to model many fishermen.
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Extraction of a common fishery with possibility of extinction, division into Exclusive Economic Zones and inherent constraints with possibility to model many fishermen by the simplest possible model.
What is the game?

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- At least 2 agents + sets to choose from + aim + interaction.
A short introduction to game theory

- A game in strategic form is defined by a triple: the set of players \( \mathcal{I} \) (usually finite or a continuum), players’ sets of available strategies \( \mathcal{S}_i \), and player’s payoff functions \( J_i \).
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  or best responds to the strategies of the others.
A short introduction to game theory cont.

- We can write it as a fixed point of the following multivalued correspondence $B : \mathcal{S} \rightarrow \mathcal{S}$, called the best response correspondence defined by

$$B_i(S) = \text{Argmax}_{s_i \in \mathcal{S}_i} J_i([s_i, S_{-i}]).$$
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- A profile $\bar{S}$ is a Nash equilibrium iff $\bar{S} \in B(\bar{S})$. 

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- A profile $\bar{S}$ is a Nash equilibrium iff $\bar{S} \in B(\bar{S})$.

- So, calculation of a Nash equilibrium requires solving a set of optimization problems in players’ strategy spaces coupled by finding a fixed point of the resulting best response correspondence in the space of strategy profiles.
A short introduction to game theory cont. 2

- What if players either do choose their strategies sequentially, there is a hierarchy or one of them has informational advantage (i.e. s/he can calculate the best response function of the other player or players). Then, instead of Nash, we consider a **Stackelberg equilibrium**.
A short introduction to game theory cont. 2

- What if players either do choose their strategies sequentially, there is a hierarchy or one of them has informational advantage (i.e. s/he can calculate the best response function of the other player or players). Then, instead of Nash, we consider a Stackelberg equilibrium.

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A profile \( \bar{S} \) is a Stackelberg equilibrium iff there exists a selection \( b_2 \in B_2 \) such that \( \bar{S}_2 \in b_2(\bar{S}_1) \).
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$S_1 \in S_1$
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- For more than two players there may be different level of hierarchy or some players at the same level: e.g. a leader and many followers playing Nash between them given the leader’s strategy (and the leader knows it and takes into its calculation). So, an optimization nested with a set of optimizations coupled by a fixed point...
We are also interested in **Pareto optimal profiles**, i.e., profiles $\bar{S}$ such that there exists no profile $S$ with

- $J_i(S) \geq J_i(\bar{S})$ for all $i$
A short introduction to game theory cont. 3

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- If the payoffs are monetary (and side payments are possible) then the most obvious Pareto optimal profile is the profile which maximizes $\sum_{i \in I} \frac{J_i(S)}{\#I}$.
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- If the payoffs are monetary (and side payments are possible) then the most obvious Pareto optimal profile is the profile which maximizes $\sum_{i \in \mathbb{I}} \frac{J_i(S)}{\# \mathbb{I}}$ (or its continuous equivalent for the continuum of players). We call such a profile the social optimum.
Example: a static fishery in an unconstrained word

- A fishery with two identical fishing firms, $S_i = \mathbb{R}_+$, linear costs of fishing $10S_i$, price dependent on the amount of fish on the market $100 - (S_1 + S_2)$.
Example: a static fishery in an unconstrained word

- A fishery with two identical fishing firms, $S_i = \mathbb{R}_+$, linear costs of fishing $10S_i$, price dependent on the amount of fish on the market $100 - (S_1 + S_2)$. So, $J_1(S_1, S_2) = (100 - (S_1 + S_2))S_1 - 10S_1$, $J_2$ analogously.
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- The best response correspondences $B_1(S_2) = \text{Argmax}(100 - (S_1 + S_2))S_1 - 10S_1 = \left\{ \frac{90 - S_2}{2} \right\}$, $B_2(S_1)$ analogously.
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- The best response correspondences $B_1(S_2) = \text{Argmax}_{S_1 \geq 0} (100 - (S_1 + S_2))S_1 - 10S_1 = \left\{ \frac{90 - S_2}{2} \right\}$, $B_2(S_1)$ analogously.

- The Nash equilibrium given by $\begin{cases} S_1 = \frac{90 - S_2}{2}, \\ S_2 = \frac{90 - S_1}{2} \end{cases}$.
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- The Nash equilibrium given by $\begin{cases} S_1 = \frac{90-S_2}{2} \\ S_2 = \frac{90-2S_1}{2} \end{cases}$. So, $S_1 = S_2 = 30$ with price 40.
Example: a static fishery in an unconstrained word

- A fishery with two identical fishing firms, \( S_i = \mathbb{R}_+ \), linear costs of fishing \( 10S_i \), price dependent on the amount of fish on the market \( 100 - (S_1 + S_2) \). So,

\[
J_1(S_1, S_2) = (100 - (S_1 + S_2))S_1 - 10S_1, \quad J_2
\]

analogously.

- The best response correspondences

\[
B_1(S_2) = \underset{S_1 \geq 0}{\text{Argmax}} (100 - (S_1 + S_2))S_1 - 10S_1 = \left\{ \frac{90 - S_2}{2} \right\},
\]

\[
B_2(S_1) \text{ analogously.}
\]

- The Nash equilibrium given by

\[
\begin{aligned}
S_1 &= \frac{90 - S_2}{2}, \\
S_2 &= \frac{90 - S_1}{2}.
\end{aligned}
\]

So,

\( S_1 = S_2 = 30 \) with price 40.

- The social optimum

\[
\begin{aligned}
\text{Argmax} & (100 - (S_1 + S_2))S_1 - 10S_1 + (100 - (S_1 + S_2))S_2 - 10S_2 \equiv ((S_1, S_2) : S_1 + S_2 = 45), \text{ price 55.}
\end{aligned}
\]
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- A fishery with two identical fishing firms, $S_i = \mathbb{R}_+$, linear costs of fishing $10S_i$, price dependent on the amount of fish on the market $100 - (S_1 + S_2)$. So,
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  \[ B_1(S_2) = \operatorname{Argmax}_{S_1 \geq 0} (100 - (S_1 + S_2))S_1 - 10S_1 = \left\{ \frac{90 - S_2}{2} \right\}, \]
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  \[
  \begin{cases} 
  S_1 = \frac{90 - S_2}{2} \\
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  \end{cases}
  \]
  So, $S_1 = S_2 = 30$ with price 40.

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  \[
  \operatorname{Argmax}_{S_1, S_2 \geq 0} (100 - (S_1 + S_2))S_1 - 10S_1 + (100 - (S_1 + S_2))S_2 - 10S_2 = \{(S_1, S_2) : S_1 + S_2 = 45\}, \]
  price 55. So, if additionally, $S_1 = S_2$, then higher profits for both.
Example: a static fishery cont.

- The Stackelberg equilibrium: After calculating $B_2(S_1)$, the leader optimizes

$$S_1 \in \text{Argmax}\left(100 - \left(S_1 + \frac{90-S_1}{2}\right)\right)S_1 - 10S_1 = 45.$$
Example: a static fishery cont.

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\[
S_1 \geq 0
\]

\[
S_2 = \frac{90 - S_1}{2} = 22.5.
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Example: a static fishery cont.

- The **Stackelberg equilibrium**: After calculating $B_2(S_1)$, the leader optimizes
  \[
  S_1 \in \text{Argmax}\left(100 - \left(S_1 + \frac{90-S_1}{2}\right)\right)S_1 - 10S_1 = 45.
  \]
  \[
  S_1 \geq 0
  \]
  \[
  S_2 = \frac{90-S_1}{2} = 22.5. \text{ Price 32.5.}
  \]
- The leader extracts more than at a Nash equilibrium and gets more payoff that at the symmetric cooperative solution and it makes the follower extract as in the symmetric cooperative solution and get less payoff than at Nash.
- That may be only the matter of informational advantage and kindly informing the follower about the resulting choice!
"The tragedy of the commons"

- Philosophically:

§ Philosophically:

- The logic of pursuing individual benefit in commons without constraints results in overexploitation (and sometimes extinction of the harvested species), and it is worse for everybody compared to the result of "mutual coercion, mutually agreed upon" but even then there is a temptation to cheat...

- § In games related to extraction of common (or interrelated) resources: the fact that the social optimum is not a Nash equilibrium and a/the Nash equilibrium (often unique) is not Pareto optimal and it yields payoffs smaller for all players than the social optimum.

- § Usually solved by enforcement: changing a game by adding a benevolent social planner – a Stackelberg leader modifying payoffs of the rest of the players by e.g. a tax in order that the previous social optimum is a Nash equilibrium given his strategy.
"The tragedy of the commons"

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- Usually solved by enforcement: changing a game by adding a benevolent social planner – a Stackelberg leader modifying payoffs of the rest of the players by e.g. a tax in order that the previous social optimum is a Nash equilibrium given his strategy.
"The tragedy of the commons" – where?

- common or interrelated fisheries,
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But they all are dynamic problems!
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Dynamic games

- **Dynamic games** are games played over time set $\mathbb{T}$, continuous or discrete, finite or infinite, with additional state variable $x \in \mathbb{X}$. 

Games
- Nash eq.
- Stackelberg eq.
- Pareto opt.
- Example-static
- The tragedy

Dynamic games
- A simple model
- Objectives
- Bellman
  - Revised sufficient condition
- Social optimum
- Nash equilibrium
  - Nash equilibria for $n$ players
  - Finite horizon truncation
- Enforcing optimality
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- Numerics
- Continuous time
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- Like in optimal control, strategies can be open loop (functions of time, initial condition fixed), feedback (function of state or state and time, initial condition arbitrary), history-dependent... depending on information structure considered.
Dynamic games – surprise 1

- Unlike in optimal control, information structure matters!
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- For some problems with a continuum of players, also a decomposition method (introduced and developed in A. Wiszniewska-Matyszkiewicz: Positivity 2002, C& C 2003, IGTR 2002, 2003, JOTA 2014) can be used and the results for open loop and feedback are equivalent in a wider class of problems (JOTA 2014).
Definition of feedback Stackelberg equilibrium is not straightforward – "the best response to every strategy of the leader" – is not a well posed problem!
Dynamic games – surprise 2

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- There are various generalizations.
Definition of feedback Stackelberg equilibrium is not straightforward – ”the best response to every strategy of the leader” – is not a well posed problem!

There are various generalizations. Some of them are not subgame perfect, some of them may result in a need to recalculate the leader’s strategy during the game (and, consequently the follower’s).
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- **Linear quadratic dynamic games (LQDG)** with linear state equation and quadratic current and terminal payoffs are most extensively studied (besides fully linear games) and have good economic interpretation.
Dynamic games cont.2

- The complexity of the problem results in the fact that we still do not know much about equilibria of dynamic games in feedback form.

- **Linear quadratic dynamic games (LQDG)** with linear state equation and quadratic current and terminal payoffs are most extensively studied (besides fully linear games) and have good economic interpretation.

- So, let’s add the inherent constraints to LQDG and we will have a nice model, with quite standard and nice results.
"The tragedy" in a simple model – the motivating example revisited

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- The set of states of the resource is \( \mathbb{R}_+ \).
"The tragedy" in a simple model – the motivating example revisited


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- **Discrete time, infinite horizon (first).**
"The tragedy" in a simple model – the motivating example revisited


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- At each time moment, player $i$ extracts amount $s_i \geq 0$, these $s_i$, in common, constitute a static profile $s$. 
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- **Constraint**: given state \( x \), the decisions have to fulfil \( s_i \in [0, cx] \).
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- The catch is sold at a **common market**.
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- Each of the players has **cost function** $\text{cost}(s_i) = fs_i + \frac{1}{2}s_i^2$.
- The catch is sold at a **common market at a price** $\text{price}(s) = A - u$,
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- Aggregate extraction influences also the state of the resource.
The model – cont.

- Increasing number of players does not mean introducing additional users,
The model – cont.

- Increasing number of players does not mean introducing additional users,
- but decomposing the decision making structure
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The model – cont.

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"The tragedy of the commons" in the dynamic context

Agnieszka Wiszniewska-Matyszkiel

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The model – cont.2

- To model this, the set of players $\mathbb{I}$ is $\{1, \ldots, n\}$ or $[0, 1]$.
The model – cont.2

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- and players are measured by the uniform normalized measure on each $\mathbb{I}$. 
The model – cont.2

- To model this, the set of players $\mathbb{I}$ is $\{1, \ldots, n\}$ or $[0, 1]$.
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  $u = \sum_{i=1}^{n} \frac{s_i}{n}$ in the case of $n$ players
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Rule of growth with fishing

$$X_i(t + 1) = (1 + \xi)X(t) - U(t)$$

(generalized later).
The model – cont.2

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- Rule of growth with fishing $X_i(t + 1) = (1 + \xi)X(t) - U(t)$ (generalized later).
- To make depletion possible, we set $c = (1 + \xi)$. 
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- To make depletion possible, we set $c = (1 + \xi)$.
- Discounting by a discount factor $\beta = \frac{1}{1+\xi}$ (the golden rule).
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Current payoff of player $i$: $P_i(s) = \text{price}(s)s_i - \text{cost}(s_i)$
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- Current payoff of player $i$: $P_i(s) = \text{price}(s) s_i - \text{cost}(s_i)$ (auxiliary notation $P(s_i, u)$ or $P(s_i, s_{\sim i})$).
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  (auxiliary notation $P(s_i, u)$ or $P(s_i, s_{\sim i})$).
- So we have a linear-quadratic dynamic game with linear state-dependent constraints on controls.
Objectives

We consider feedback strategies – choices of decisions as functions of state, $S_i(x)$.
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- The objective—payoff function of player $i$ is

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J_i(S) = \sum_{t=0}^{\infty} P_i(S(X(t)))\beta^t \text{ (for feedback controls).}
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- We want to calculate the social optima,
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- and Nash equilibria,
  - i.e. profiles at which each player maximizes their payoff given strategies of remaining players.
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- Calculation of both require solving dynamic optimization problems.
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- Calculation of both require solving dynamic optimization problems.

- In the case of Nash equilibrium, a set of dynamic optimization problems.
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- The objective—payoff function of player $i$ is
  \[ J_i(S) = \sum_{t=0}^{\infty} P_i(S(X(t)))\beta^t \] (for feedback controls).
- We want to calculate the social optima,
  - i.e. profiles which maximize aggregate payoff;
- and Nash equilibria,
  - i.e. profiles at which each player maximizes their payoff given strategies of remaining players.
- Calculation of both require solving dynamic optimization problems.
- In the case of Nash equilibrium, a set of dynamic optimization problems coupled by finding a fixed point in the space of feedback strategy profiles.
Standard infinite horizon Bellman sufficient condition

- For a dynamic optimization problem
  - maximize $J(\tilde{t}, \bar{x}, U) = \sum_{t=\tilde{t}}^{\infty} g(X(t), U(X(t), t), t) \delta^{t-\tilde{t}}$, where $\delta$ is the discount factor.
Standard infinite horizon Bellman sufficient condition

- For a dynamic optimization problem
  - maximize $J(\tilde{t}, \tilde{x}, U) = \sum_{t=\tilde{t}}^{\infty} g(X(t), U(X(t), t), t)\delta^{t-\tilde{t}}$,
  - for $X$ defined by $X(t + 1) = f(X(t), U(X(t), t), t)$
Standard infinite horizon Bellman sufficient condition

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  - for $X$ defined by $X(t + 1) = f(X(t), U(X(t), t), t)$ with initial condition $X(\bar{t}) = \bar{x}$.
  - We assume $J(\tilde{t}, \bar{x}, U)$ is always well defined, although it can be $-\infty$. 
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- If a function $V : X \times \mathbb{N} \rightarrow \mathbb{R}$ fulfils the Bellman equation:
  - $(BE) \ V(x, t) = \sup_{u \in U} g(x, u, t) + \delta \ V(f(x, u, t), t + 1)$
Standard infinite horizon Bellman sufficient condition

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  - maximize $J(\bar{t}, \bar{x}, U) = \sum_{t=\bar{t}}^{\infty} g(X(t), U(X(t), t), t)\delta^{t-\bar{t}}$,
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- If a function $V : \mathbb{X} \times \mathbb{N} \rightarrow \mathbb{R}$ fulfils the Bellman equation:
  - (BE) $V(x, t) = \sup_{u \in U} g(x, u, t) + \delta V(f(x, u, t), t + 1)$ with the terminal condition:
  - (TC) for every trajectory $X$, $\limsup_{t \rightarrow \infty} V(X(t), t) \delta^t = 0$
Standard infinite horizon Bellman sufficient condition

For a dynamic optimization problem

- maximize $J(\tilde{t}, \bar{x}, U) = \sum_{t=\tilde{t}}^{\infty} g(X(t), U(X(t), t), t) \delta^{t-\tilde{t}}$, 
- for $X$ defined by $X(t+1) = f(X(t), U(X(t), t), t)$ with initial condition $X(\tilde{t}) = \bar{x}$.
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Standard infinite horizon Bellman sufficient condition

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  - maximize $J(\bar{t}, \bar{x}, U) = \sum_{t=\bar{t}}^{\infty} g(X(t), U(X(t), t), t)\delta^{t-\bar{t}}$,
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- If a function $V : \mathbb{X} \times \mathbb{N} \rightarrow \mathbb{R}$ fulfils the Bellman equation:
  - (BE) $V(x, t) = \sup_{u \in \mathbb{U}} g(x, u, t) + \delta \cdot V(f(x, u, t), t + 1)$ with the terminal condition:
  - (TC) for every trajectory $X$, $\limsup_{t \to \infty} V(X(t), t) \delta^t = 0$

- then $V$ is the value function of the dynamic optimization problem, while any selection from the Argmax of the rhs. of the (BE) is an optimal control.
Revised sufficient condition

- (TC) can be replaced by
- (TC’)
  - (a) for every admissible trajectory $X$
    \[
    \limsup_{t \to \infty} V(X(t), t) \delta^t \leq 0
    \]
Revised sufficient condition

- (TC) can be replaced by
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  - (a) for every admissible trajectory $X$
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  - (b) and if $\limsup_{t \to \infty} V(X(t), t) \delta^t < 0$,
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Revised sufficient condition

- \((\text{TC})\) can be replaced by
- \((\text{TC}')\)
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    \]
  - (b) and if \(\limsup_{t \to \infty} V(X(t), t) \delta^t < 0\), then \(J(t, x, U) = -\infty\)
    for every \(U\) such that trajectory \(X\) is corresponding to it.


- (b) is necessary!
Revised sufficient condition

- **(TC)** can be replaced by
- **(TC’)**
  - **(a)** for every admissible trajectory $X$
    $$\limsup_{t \to \infty} V(X(t), t) \delta^t \leq 0$$
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- **(b)** is necessary! and **(a)** is also necessary under very week condition
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Social optimum

- The solution is symmetric.
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- We solve the problem assuming quadratic value function $V(x) = hx^2 + gx + k$ (by undetermined coefficient method).
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- Of all those solutions, only \( V(x) = hx^2 + gx \) with negative \( h \) solves \((BE)\) on the whole domain!
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- Of all those solutions, only $V(x) = hx^2 + gx$ with negative $h$ solves (BE) on the whole domain!
- For this $V$, the optimum of the rhs. of (BE) is $\xi x$,
- which results in constant state trajectory.
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- Of all those solutions, only \( V(x) = hx^2 + gx \) with negative \( h \) solves (BE) on the whole domain!
- For this \( V \), the optimum of the rhs. of (BE) is \( \xi x \), which results in constant state trajectory.
- But \( V(x) = hx^2 + gx \) with negative \( h \) which solves (BE) (and it is the only quadratic solution of it) is not the value function.
Counterexample!!!

- The only quadratic solution of (BE) is not the value function!
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- The Bellman equation, if we neglect constraints, has also continuum of linear solutions, \( gx + \hat{k} \) for arbitrary \( g \).
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- Of course, (TC’) is not fulfilled.
- The Bellman equation, if we neglect constraints, has also continuum of linear solutions, $gx + \hat{k}$ for arbitrary $g$.
- The solution corresponding to the quadratic $V$ is $\xi x$. It guarantees sustainability – so it is not enough to check (TC) along the trajectory corresponding to maximizer of rhs of (BE), as it is sometimes done.

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- \(g = 0\) does not solve \((\text{BE})\) for small \(x\).
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- The solution corresponding to the quadratic \(V\) is \(\xi x\). It guarantees sustainability – so it is not enough to check \((TC)\) along the trajectory corresponding to maximizer of rhs of \((BE)\), as it is sometimes done.
- The solutions with nonzero \(g\) also violate \((TC')\).
- \(g = 0\) does not solve \((BE)\) for small \(x\).
- There is also a solution with the only piecewise quadratic \(V\) that fulfils both \((BE)\) and \((TC)\).
Social optimum cont.

**Theorem 1**

**(a)** The value function per player is

\[
\bar{V}(x) = \begin{cases} 
\hat{g} \cdot x + \frac{\hat{n}}{2} \cdot x^2 & \text{if } x \in (0, \frac{\hat{S}}{\xi}), \\
\tilde{k} & \text{otherwise,}
\end{cases}
\]
social optimum cont.

**Theorem 1**

(a) The value function per player is

$$\bar{V}(x) = \begin{cases} \hat{g} \cdot x + \frac{\hat{h}}{2} \cdot x^2 & \text{if } x \in (0, \frac{\hat{s}}{\xi}), \\ \tilde{k} & \text{otherwise,} \end{cases}$$

for $\hat{s} = \frac{A-f}{3}$, $\hat{h} = -3 \xi (1 + \xi)$, $\hat{g} = (A - f)(1 + \xi)$, and $\tilde{k} = \frac{(A-f)^2(1+\xi)}{6\xi}$. 
Social optimum cont.

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and it is independent of the number of players (both \( n \geq 1 \) and continuum).
Social optimum cont.

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![Figure: Value function per player for social optimum](image)
Social optimum cont. 2

**Theorem 1 cont. (b)** A profile defined by

$$\hat{S}_i^{SO}(x) = \begin{cases} \xi x, & x \in (0, \hat{s}), \\ \hat{s} & \text{otherwise}, \end{cases}$$

is the unique social optimum both for $n$ players and the continuum of players.

*Figure: Strategy of each player at social optimum*
Social optimum cont. 2

**Theorem 1 cont. (b)** A profile defined by

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**Figure:** Strategy of each player at social optimum

For piecewise defined \( \bar{V} \) and \( \mathbf{s} \), the Bellman equation has to be checked again!
Nash equilibrium for continuum of players

- Different method of calculation
Nash equilibrium for continuum of players

- Different method of calculation – a decomposition method (a dynamic game decomposed into a sequence of static games).
Nash equilibrium for continuum of players

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Figure: Strategy of each player at the Nash equilibria
Nash equilibrium for continuum of players

- Different method of calculation – a decomposition method (a dynamic game decomposed into a sequence of static games).

Figure: Strategy of each player at the Nash equilibria

- Exploitation many times larger than at the social optimum.
Nash equilibrium for continuum of players

**Theorem 2 (a)** The profile defined by

\[ \hat{S}_i^{\text{NE}}(x) = \begin{cases} \frac{A-f}{2} & \text{otherwise}, \\ (1 + \xi) x & \text{for } x \leq \hat{x}_1, \end{cases} \]

for \( \hat{x}_1 = \frac{A-f}{2(1+\xi)} \), is the only feedback Nash equilibrium profile (up to measure equivalence).
Nash equilibrium for continuum of players

**Theorem 2 cont. (b)** The function defined by

\[
\bar{V}_i^{\text{NE}}(x) = \begin{cases} 
P_{\text{depl}}(x), & \text{for } x \leq \hat{x}_1 \\
\frac{\sum_{k=1}^{N} (A-f)^2 \beta^{k-1}}{8} + \beta^N P_{\text{depl}} \left( (1 + \xi)^N x - \frac{(A-f) \sum_{k=1}^{N} \beta^{k-1}}{2} \right) & \text{for } x \in (\hat{x}_N, \hat{x}_1) \\
\frac{(A-f)^2}{8} \cdot \frac{(1+\xi)}{\xi} & \text{otherwise},
\end{cases}
\]

for \( P_{\text{depl}}(x) = P(1 + \xi) x, (1 + \xi) x \) (payoff resulting from immediate depletion of the resource) and \( \hat{x}_N = \frac{A-f}{2} \sum_{k=1}^{N} \beta^k \) for \( N \geq 1 \), is the value function for optimization problem for the continuum of players game.

**Figure:** Value function at Nash equilibrium for continuum of players
Value function through a magnifying glass

Figure: Value function at Nash equilibrium for continuum of players – zoomed view
Nash equilibrium for continuum of players cont2.

**Theorem 2 cont.** (c) For \( x \in (\hat{x}_N, \hat{x}_{N+1}] \) with \( \hat{x}_0 = 0 \), the resource will be depleted/extracted in \( N + 1 \) stages, while for \( x \geq \hat{x}_\infty = \lim_{N \to \infty} \hat{x}_N \), the resource will never be depleted.

![Graph showing the number of time moments to resource exhaustion at Nash equilibrium for continuum of players](image-url)

**Figure:** Number of time moments to resource exhaustion at Nash equilibrium for continuum of players
Nash equilibria for $n$ players

- For $n$ players, a similar value function to that for continuum of players, with number of stages to depletion nonstrictly increasing as $x$ increases, can be expected.
Nash equilibria for $n$ players

- For $n$ players, a similar value function to that for continuum of players, with number of stages to depletion nonstrictly increasing as $x$ increases, can be expected.
- However, it is not possible for analogous form of equilibrium strategies, piecewise linear with two intervals.
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- The only thing we were able to prove (with reasonable length of proof) is that the number of pieces in both equilibrium and value function is greater than two.
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- Any attempt to determine the symmetric solution (with possibly infinitely many "switches") assuming continuity (with respect to state) of: the value functions
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- Any attempt to determine the symmetric solution (with possibly infinitely many ”switches”) assuming continuity (with respect to state) of: the value functions or the equilibrium strategies or the rhs of the Bellman equation along the optimal equilibrium strategy...
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- Any attempt to determine the symmetric solution (with possibly infinitely many "switches") assuming continuity (with respect to state) of: the value functions or the equilibrium strategies or the rhs of the Bellman equation along the optimal equilibrium strategy or another function related to depletion of resources was unsuccessful.
Nash equilibria for $n$ players

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Nash equilibria for $n$ players continued

- Let us skip the continuity assumption
Nash equilibria for $n$ players continued

- Let us skip the continuity assumption and allow the Nash equilibrium strategies to be
Nash equilibria for \( n \) players continued

- Let us skip the continuity assumption and allow the Nash equilibrium strategies to be
  - discontinuous at the points at which the number of time moments to depletion changes;
Nash equilibria for \( n \) players continued

- Let us skip the continuity assumption and allow the Nash equilibrium strategies to be
  - **discontinuous** at the points at which the number of time moments to depletion changes;
  - constant strategies and value function for \( x \) above some level;
Nash equilibria for $n$ players continued

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  - discontinuous at the points at which the number of time moments to depletion changes;
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  - proving that requires more compound tools than the continuum of players Nash equilibrium and any social optimum;
Nash equilibria for \( n \) players continued

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  - discontinuous at the points at which the number of time moments to depletion changes;
  - constant strategies and value function for \( x \) above some level;
  - proving that requires more compound tools than the continuum of players Nash equilibrium and any social optimum;
  - a symmetric piecewise linear Nash equilibrium, if it exists, is discontinuous (and we can state its general form up to location the points of discontinuity and checking the Bellman inclusion for the discontinuous, non-quasi concave function at the rhs.) and
Nash equilibria for $n$ players continued

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  - discontinuous at the points at which the number of time moments to depletion changes;
  - constant strategies and value function for $x$ above some level;
  - proving that requires more compound tools than the continuum of players Nash equilibrium and any social optimum;
  - a symmetric piecewise linear Nash equilibrium, if it exists, is discontinuous (and we can state its general form up to location the points of discontinuity and checking the Bellman inclusion for the discontinuous, non-quasi concave function at the rhs.) and
  - it is a limit of Nash equilibria for finite horizon truncations of the game
Nash equilibria for \( n \) players continued

- Let us skip the continuity assumption and allow the Nash equilibrium strategies to be
  - discontinuous at the points at which the number of time moments to depletion changes;
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  - it is a limit of Nash equilibria for finite horizon truncations of the game
  - and the irregularity is inherited from finite horizon truncations of the game.
Nash equilibria in the $n = 2$ players truncated game

- So, we analyse truncations
Nash equilibria in the \( n = 2 \) players truncated game

Nash equilibria in the $n = 2$ players truncated game


- In the **two stage truncation** of the game
Nash equilibria in the \( n = 2 \) players truncated game

- In the two stage truncation of the game: a continuum of...
Nash equilibria in the $n = 2$ players truncated game


- In the two stage truncation of the game: a continuum of discontinuous symmetric equilibria
Nash equilibria in the $n = 2$ players truncated game


- In the two stage truncation of the game: a continuum of discontinuous symmetric equilibria and no continuous symmetric equilibrium!
Nash equilibria in the $n = 2$ players truncated game

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**Figure:** Two stage truncation of the game

(a) two symmetric Nash equilibria
Nash equilibria in the $n = 2$ players truncated game


- In the two stage truncation of the game: a continuum of discontinuous symmetric equilibria and no continuous symmetric equilibrium!

**Figure**: Two stage truncation of the game

(a) two symmetric Nash equilibria

(b) two symmetric Nash equilibria—zoomed view
Nash equilibria in the $n = 2$ players truncated game

**Figure:** Two stage truncation of the game—the value functions at two symmetric Nash equilibria
Enforcing social optimality by a tax or tax-subsidy system

- Introduction of a regulatory tax
Enforcing social optimality by a tax or tax-subsidy system

- Introduction of a regulatory tax
  \[ P(s_i, s_{\sim i}) \sim P(s_i, s_{\sim i}) - T(s_i, x) \]
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- The rate of linear tax
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Enforcing social optimality by a tax or tax-subsidy system

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- The rate of **linear tax** \( T(s_i, x) = \tau(x)s_i \) enforcing social optimality in the **continuum of players** game is given by

  \[
  \tau(x) = \begin{cases} 
  A - f - 2\xi x & \text{if } x \leq \frac{A - f}{3\xi}, \\
  \frac{A - f}{3} & \text{otherwise}. 
  \end{cases}
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Enforcing social optimality by a tax or tax-subsidy system

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**Figure: Rate of tax enforcing social optimality for continuum of players**
Enforcing social optimality by a tax – cont.

- Variable tax rate?

Enforcing optimality by a tax – cont.

- Variable tax rate?

It is not a problem, since:

- if from time 0 on the regulator chooses the tax rate $\tau$, then the state is constantly $x_0$ and the resulting Nash equilibrium is equal to social optimum in the initial problem;

- generally, if instead of “tax” we use the term “environmental levy”, then increasing the levy as the state of the environment deteriorates seems justified.

The resulting tax paid is: $\text{State (x)} \times 10^4$

<table>
<thead>
<tr>
<th>State (x)</th>
<th>Tax enforcing social optimality</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1.5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>2.5</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

Figure: Tax enforcing social optimality for continuum of players
Enforcing social optimality by a tax – cont.

- Variable tax rate? It is not a problem
Enforcing social optimality by a tax – cont.

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- The resulting tax paid is

![Graph](image)

**Figure:** Tax enforcing social optimality for continuum of players
Enforcing social optimality by a tax – cont.

- If we consider a **tax-subsidy system** with
  \[ T(s_i, x) = \tau(x)(s_i - \bar{S}_i^{SO}) \]
Enforcing social optimality by a tax – cont.

- If we consider a **tax-subsidy system** with
  \[ T(s_i, x) = \tau(x)(s_i - \bar{S}_i^{SO}) \] - then the results are equivalent

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"The tragedy of the commons" in the dynamic context

Agnieszka Wiszniewska-Matyszkiewicz

Games
- Nash eq.
- Stackelberg eq.
- Pareto opt.
- Example-static
- The tragedy

Dynamic games
- A simple model
- Objectives

Bellman
- Revised sufficient condition

Social optimum
- Nash equilibrium
- Nash equilibria for \( n \) players
- Finite horizon truncation

Enforcing optimality
- Carrying capacity
- Numerics
- Continuous time

Conclusions
Enforcing social optimality by a tax – cont.

- If we consider a tax-subsidy system with 
  \[ T(s_i, x) = \tau(x)(s_i - \bar{S}_i^{SO}) \] 
  – then the results are equivalent (i.e. the same \( \tau \) enforces \( \bar{S}_i^{SO} \),
Enforcing social optimality by a tax – cont.

- If we consider a tax-subsidy system with
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- If we consider the tax rate \( \tau \) calculated for the continuum of players.

"The tragedy of the commons" in the dynamic context

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Dynamic games
A simple model
Objectives
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- So, the continuum of players model helped us to solve the problem of enforcement for \( n \) players although we are not able to calculate the Nash equilibrium for \( n \) players.
Extensions of the model and introducing carrying capacity

- All the above results remain valid
Extensions of the model and introducing carrying capacity

- All the above results remain valid if we appropriately modify the dynamics of the state above in order to take into account the carrying capacity of the environment.
Be careful with numerics!

- Solving (BE) numerically is costly.
Be careful with numerics!

- Solving (BE) numerically is costly.
- A class of optimal control problems (i.e. \( n = 1 \)), analogous to our social optimality but for a whole interval of possible discount factors (a slightly more impatient decision makers): for a candidate \( V^f \) for the value function calculated analogously as in Theorem 1 and a control \( S^f \) from the rhs of the (BE) with \( V^f \), for every \( \epsilon > 0 \), there is a discount factor close to the golden rule such that the Bellman equation is fulfilled everywhere besides an \( \epsilon \)-neighbourhood of 0, while \( S^f \) is far from the optimal control while \( V^f \) from the value function on the set of all reasonable states (i.e. below \( \hat{s} \)).

- Nested induction (backward and forward) plus concave analysis needed to derive the optimal control analytically – piecewise linear with infinitely many pieces.

We considered a model from similar class in continuous time to model a cryptocurrency mining game.
Continuous time

- We considered a model from similar class in continuous time to model a cryptocurrency mining game.
- General theory for such problems still not developed (viscosity solutions for infinite horizon, sufficiency and necessity, etc.)
Conclusions ”The tragedy of the commons” in dynamic context

- To model most of the tragedy of the commons problems, tools of dynamic games are required, especially feedback Nash and Stackelberg equilibria.
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- Problems of feedback Nash equilibria require solving a set of coupled parametrized dynamic optimization problems, with strategies of the others as parameters.
- Problems of feedback Stackelberg equilibria are even more complicated.
- Only few classes of such games have been solved and some proofs are still incomplete.
- Models lack realistic constraints.
- Adding even very inherent constraints can change the solutions drastically, with several surprises.
Conclusions – LQ games with constraints

- After imposing natural constraints (by the amount of resource available)
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- After imposing **natural constraints** (by the amount of resource available) and making **exhaustion possible**, ...
Conclusions – LQ games with constraints

- After imposing natural constraints (by the amount of resource available) and making exhaustion possible, a linear quadratic game of resource extraction yields results which are contrary to standard results in LQ dynamic games.
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- Social optimum for this problem is a simple counterexample to the correctness of commonly used skipping checking terminal condition — the only ”nice” solution of (BE) is not the value function), which it started a research on necessity of the terminal condition.
Nash equilibrium for the continuum of players case is piecewise linear with value function piecewise quadratic with infinitely many pieces, non-monotone, non-differentiable.
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The continuum of players game helps to find solutions for \( n \) players games!
Thank you for your attention!