The Szlenk index and asymptotic geometry of Banach spaces

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(B) it provides a Banach-space theoretic analogue of the Cantor–Bendixson index from classical topology;

(C) the Szlenk power type is a quantity which carries information about asymptotic structure/geometry of a given Banach space.

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- $f_n \to f$ with respect to the norm topology if and only if (f_n) converges to f **uniformly** on B_X (equivalently, on any bounded subset of X)

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Note: The weak* topology is strictly weaker than the norm topology unless X is finite-dimensional.

A simple question

Let X be a Banach space and B_{X^*} be the dual unit ball. When is it possible to find nonempty open subsets of B_{X^*} in the relative weak* topology with arbitrarily small diameter? In other words, we want that for each $\varepsilon > 0$ there exists a weak* open set $U \subset X^*$ with

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- $X = \ell_1$: NO. Every nonempty relatively weak* open subset of B_{ℓ_∞} has diameter 2.
- $X = C[0,1]$: NO. Every measure μ in the unit ball of $C[0,1]^* = \mathcal{M}[0,1]$ can be weak* approximated by a sequence of measures all being at distance at least 1 from μ.

- For example, for the Lebesgue measure λ we have

$$\mu_n := \frac{1}{n}\sum_{j=1}^{n}\delta_{j/n} \xrightarrow{w*} \lambda$$

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- $X = c_0$: YES. Although every weak* neighbourhood of zero in B_{ℓ_1} has diameter 2, we can consider elements $f \in B_{\ell_1}$ with $\|f\| > 1 - \frac{\varepsilon}{2}$ which have weak* neighbourhoods in B_{ℓ_1} of diameter smaller than ε.

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- $X = c_0$: YES. Although every weak* neighbourhood of zero in B_{ℓ_1} has diameter 2, we can consider elements $f \in B_{\ell_1}$ with $\|f\| > 1 - \frac{\varepsilon}{2}$ which have weak* neighbourhoods in B_{ℓ_1} of diameter smaller than ε. Indeed, we can take a weak* neighbourhood U of f in such a way that there is $N \in \mathbb{N}$ such that for every $g = (\eta_n) \in U$ all the coordinates $\eta_1, \ldots, \eta_N$ 'almost' agree with the corresponding coordinates of f and $\sum_{j=1}^{N} |\eta_j| > 1 - \frac{\varepsilon}{2}$. Then, for $g \in U \cap B_{\ell_1}$ we have $\sum_{j>N} |\eta_j| < \frac{\varepsilon}{2}$ and hence $\|g - h\|_1 < \varepsilon$ for all $g, h \in U \cap B_{\ell_1}$.

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I. Namioka, R.R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J. **42** (1975), 735–750.

Define $D = \{x^* \in B_{X^*} : \|x^*\| \leq \varepsilon/2\}$ and notice that the whole of B_{X^*} can be covered by countably many translations of D. By the Baire Category Theorem, one of them contains a nonempty weak* open set.

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Theorem (Namioka and Phelps)

If X is a Banach space with X^* separable, then every nonempty bounded subset B of X^* contains nonempty weak* slices $S(B; x, \alpha)$ of arbitrarily small diameter, where

$$S(B; x, \alpha) = \left\{x^* \in B : x^*(x) \geq \sup_{z^* \in B} z^*(x) - \alpha\right\}.$$

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Observe that if K is not scattered, then $B_{C(K)^*}$ does not contain nonempty relatively weak* open sets of arbitrarily small diameter. Indeed, if $p \in K$ is not isolated and $(p_n) \subset K \setminus \{p\}$ converges to p, then

$$\delta_{p_n} \xrightarrow{w*} \delta_p \quad \text{and} \quad \|\delta_{p_n} - \delta_p\| = 2.$$

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$$s_\varepsilon(K) = \Big\{x^* \in K \colon \mathrm{diam}(V \cap K) > \varepsilon \text{ for every weak}^* \text{ open neighborhood of } x^*\Big\},$$

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and transfinite derivations by

$$s_\varepsilon^0(K) = K, \;\; s_\varepsilon^{\xi+1}(K) = s_\varepsilon(s_\varepsilon^\xi(K))$$

and

$$s_\varepsilon^\xi(K) = \bigcap_{\zeta<\xi} s_\varepsilon^\zeta(K)$$

for ξ being a limit ordinal.

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Remark (follows from the Namioka–Phelps theorem)

The Szlenk index $Sz(X)$ is properly defined if and only if X is an Asplund space, that is, the dual of each separable subspace of X is separable.

Let X be a separable Banach space. Then $Sz(X)$ is well-defined if and only if X^* is separable, and then we must have $Sz(X) < \omega_1$. Indeed, just observe that $(s_\varepsilon^\xi B_{X^*})_\xi$ is a strictly decreasing family of weak* closed subsets of the separable (Polish) space (B_{X^*}, w^*).

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Then

$$0 \in s_1^\alpha B_{X_\alpha^*} \quad \text{for every } \alpha < \omega_1.$$

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J. Bourgain, *On separable Banach spaces universal for all separable reflexive spaces*, Proc. Amer. Math. Soc. **79** (1980), 241–246.

Theorem (Bourgain)

If Z is any Banach universal for all separable reflexive Banach spaces (in the above sense), then Z contains an isomorphic copy of $C[0,1]$.

- $Sz(X) = 1$ if and only if $\dim X < \infty$;

The Szlenk index

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- $Sz(X \hat{\otimes}_\varepsilon Y) = \omega$ provided that $\max\{Sz(X), Sz(Y)\} = \omega$ (Causey, 2013); for example, $Sz(\mathcal{K}(\ell_p, \ell_q)) = \omega$ for any $1 < p, q < \infty$;

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Remark

The Szlenk index (if correctly defined) is always of the form ω^α. Note also that, by a compactness argument, the ε-Szlenk indices cannot be limit ordinals.

G. Lancien, *A survey on the Szlenk index and some of its applications*, Rev. R. Acad. Cien. Serie A. Mat. **100** (2006), 209–235.

Observation (Lancien)

For any Banach space X and any $\varepsilon, \eta > 0$ we have

$$s_{\varepsilon\eta}^{\alpha \cdot Sz(X,\eta)}(B_{X^*}) \subseteq s_{\varepsilon}^{\alpha}(B_{X^*})$$

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If $Sz(X) = \omega$, then all the ε-indices $Sz(X, \varepsilon)$ are just natural numbers, as they are at most ω and cannot be limit ordinals. Hence, if $Sz(X) = \omega$, we have a subadditive function

$$(0, \infty) \ni t \longmapsto \log Sz(X, e^{-t}) =: \phi(t).$$

By the classical Fekete's lemma, there exists a finite limit

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = \inf_{t\geq\alpha} \frac{\phi(t)}{t} \quad (\alpha > 0).$$

Definition

Let X be a Banach space with $Sz(X) = \omega$. We define its *Szlenk power type* $\mathrm{p}(X) \in [1, \infty)$ by the formulas:

$$\begin{aligned} \mathrm{p}(X) &= \lim_{\varepsilon\to 0+} \frac{\log Sz(X,\varepsilon)}{|\log \varepsilon|} \\ &= \inf\Big\{q \geq 1 \colon \sup_{\varepsilon\in(0,1)} \varepsilon^q \cdot Sz(X,\varepsilon) < \infty\Big\}. \end{aligned}$$

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Moreover, if X and Y are isomorphic of Szlenk index ω, and d is the Banach–Mazur distance between X and Y, then $Sz(X, d\varepsilon) \leq Sz(Y, \varepsilon)$, whence

$$\mathrm{p}(Y) = \lim_{\varepsilon \to 0+} \frac{\log Sz(Y, \varepsilon)}{|\log \varepsilon|} \geq \lim_{\varepsilon \to 0+} \frac{\log Sz(X, d\varepsilon)}{|\log \varepsilon|} = \mathrm{p}(X).$$

We can interpret the Szlenk power type as follows: It is the optimal exponent which describes the rate of cutting out the dual unit ball by iterates of Szlenk derivations.

$$\mathrm{p}(X) = \inf\Big\{ q \geq 1\colon \sup_{\varepsilon\in(0,1)} \varepsilon^q \cdot Sz(X,\varepsilon) < \infty \Big\}$$

In general, the infimum may not be attained. However, for every $\delta > 0$ there exists $C < \infty$ such that

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Thus, the Szlenk power type is an isomorphic invariant.

Detour to Tsirelson's space

No simple structural theory!

A long standing open problem going back to Banach's book:

Does every infinite-dimensional Banach space contain a subspace isomorphic to either c_0 *or* ℓ_p *for some* $1 \leq p < \infty$*?*

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The Figiel–Johnson version

The dual space $\mathfrak{T}^*$ enjoys the same properties, it is the completion of c_{00} under a norm $\|\cdot\|$ defined implicitely as

$$\|\xi\| = \|\xi\|_0 \vee \frac{1}{2} \sup \left\{ \sum_{j=1}^{m} \|I_j \xi\| \colon m < I_1 < I_2 < \ldots < I_m \right\}.$$

Recall that the Szlenk power type $\mathrm{p}(X)$ is well-defined for any Banach space X with $Sz(X) = \omega$ and then $\mathrm{p}(X) \in [1, \infty)$.

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- $\mathrm{p}(\mathfrak{T}(c_0)) = 1$ (Draga and K., J. Funct. Anal. 2016). An example showing that the infimum in the definition of the Szlenk index may not be attained;
- $\mathrm{p}(X \hat{\otimes}_\varepsilon Y) = \max\{\mathrm{p}(X), \mathrm{p}(Y)\}$ whenever $Sz(X), Sz(Y) \leq \omega$ (Draga and K., Proc. Amer. Math. Soc. 2017). In particular, $\mathrm{p}(\mathcal{K}(\ell_p, \ell_q)) = \max\{p, \frac{q}{q-1}\}$.

R. Causey, *Estimation of the Szlenk index of Banach spaces via Schreier spaces*

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Theorem (Causey)

Let X and Y be nonzero separable Banach spaces with $\mathrm{Sz}(X)$ and $\mathrm{Sz}(Y)$ at most ω. Then

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$$\mathrm{Sz}(X \hat{\otimes}_\varepsilon Y) = \max\{\mathrm{Sz}(X), \mathrm{Sz}(Y)\}.$$

Consequently, it makes sense to ask about the Szlenk power of the injective tensor product of two Banach spaces with Szlenk index ω.

V.D. Milman, *Geometric theory of Banach spaces. II. Geometry of the unit ball* (Russian), Uspehi Mat. Nauk **26** (1971), 73–149.

In 1971, V.D. Milman initiated the study of asymptotic geometry of Banach spaces by introducing the notions of moduli of asymptotic smoothness/convexity.

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In 1971, V.D. Milman initiated the study of asymptotic geometry of Banach spaces by introducing the notions of moduli of asymptotic smoothness/convexity.

Definition

- $\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \|x + ty\| - 1$ (the modulus of asymptotic uniform smoothness);
- $\overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\dim(X/Y)<\infty} \inf_{y \in S_Y} \|x + ty\| - 1$ (the modulus of asymptotic uniform convexity);
- $\overline{\delta}_X^*(t) = \inf_{x^* \in S_{X^*}} \sup_E \inf_{y^* \in S_E} \|x^* + ty^*\| - 1$, where E runs through all weak*-closed subspaces of X^* with finite codimension
(the modulus of weak* asymptotic uniform convexity).

Example. If F_n are finite-dimensional ($n \in \mathbb{N}$) and $X = (\bigoplus_{n=1}^{\infty} F_n)_p$, then $\overline{\rho}_X(t) = \overline{\delta}_X(t) = (1 + t^p)^{1/p} - 1$.

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Definition

A Banach space X is called *asymptotically uniformly smooth* (*convex*) provided that

$$\overline{\rho}_X(t) = o(t) \text{ as } t \to 0 \qquad (\overline{\delta}_X(t) > 0 \text{ for each } t > 0).$$

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For instance, ℓ_1 is asymptotically uniformly convex although it is not uniformly convexifiable (as it is not superreflexive; recall Enflo's theorem).

H. Knaust, E. Odell, Th. Schlumprecht, *On asymptotic structure, the Szlenk index and UKK properties in Banach spaces*, Positivity **3** (1999)

G. Godefroy, N.J. Kalton, G. Lancien, *Szlenk indices and uniform homeomorphisms*, Trans. Amer. Math. Soc. **353** (2001), 3895–3918.

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Theorem (G. Godefroy, N.J. Kalton, G. Lancien)

If X is a separable Banach space with $Sz(X) \leq \omega$, then

$$p(X) = \inf\Big\{p \geq 1\colon \exists_{\text{equiv. norm } |\cdot| \text{ on } X}\ \exists_{c>0}\ \forall_{t>0}\ \overline{\delta}^*_{|\cdot|}(t) \geq ct^p\Big\}$$
$$= \inf\Big\{q \geq 1\colon \exists_{\text{equiv. norm } |\cdot| \text{ on } X}\ \exists_{C>0}\ \forall_{t>0}\ \overline{\rho}_{|\cdot|}(t) \leq Ct^p, \text{ where } p^{-1} + q^{-1} = 1\Big\}.$$

V.D. Milman, N. Tomczak-Jaegermann (Contemp. Math. 1993)

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Definition

Let X be a Banach space and $n \in \mathbb{N}$. We say that a sequence $(e_j)_{j=1}^n \in \mathcal{M}_n$ is an *element of the* n^{th} *asymptotic structure of* X, and then we write $(e_j)_{j=1}^n \in \{X\}_n$, provided that

$$\forall \varepsilon > 0 \; \forall Y_1 \in \operatorname{cof}(X) \; \exists y_1 \in S_{Y_1} \; \ldots \; \forall Y_n \in \operatorname{cof}(X) \; \exists y_n \in S_{Y_n}$$
$$d_b((y_j)_{j=1}^n, (e_j)_{j=1}^n) < 1 + \varepsilon.$$

In other words, $(e_j)_{j=1}^n \in \{X\}_n$ if and only if for every $\delta > 0$ Player II has a winning strategy in the $\mathcal{A}_\delta$-game, where $\mathcal{A}_\delta$ is the ball in $\mathcal{M}_n$ with center $(e_j)_{j=1}^n$ and radius δ.

Although an infinite-dimensional Banach space may not contain c_0 and ℓ_p, for any $1 \leq p < \infty$, it must do so 'asymptotically'. There is a famous theorem by Krivine (1976) which can be formulated as follows:

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For any infinite-dimensional Banach space X, there exists $1 \leq p \leq \infty$ such that

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Therefore, one can naturally associate with any infinite-dimensional Banach space X its *Krivine spectrum* defined as the set of all corresponding p's.

E. Odell, Th. Schlumprecht, *Embedding into Banach spaces with finite dimensional decompositions*, Rev. R. Acad. Cien. Serie A. Mat. **100** (2006), 295–323.

Theorem (Odell and Schlumprecht)

Let X be a Banach space with X^* separable. Then, the following conditions are equivalent:

(i) $\mathrm{Sz}(X) = \omega$;

(ii) there exists $q > 1$ and $K < \infty$ so that for every sequence $(e_i)_{i=1}^{\infty} \in \{X\}_n$ (the n^{th} asymptotic structure of X) and every sequence of scalars $(a_i)_{i=1}^{n}$ we have

$$\Big\|\sum_{i=1}^{n} a_i e_i\Big\| \leq K\Big(\sum_{i=1}^{n} |a_i|^q\Big)^{1/q}.$$

Theorem (Draga and K.)

In fact, the said q (occuring in upper ℓ_q-estimates) may be taken to be arbitrarily close to the conjugate of the Szlenk power type $\mathrm{p}(X)$.

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The lower bound of Krivine's spectrum is at least equal to $\mathrm{p}(X)'$.