A Topological Ramsey Theorem

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Ramsey’s Theorem

For any $r, n \in \omega$ and any $f : [\omega]^r \to n$ there is $H \subseteq \omega$ and $i < n$ such that

$$f(x) = i \text{ for all } x \in [H]^r$$

$H$ is homogeneous for $f$

I.e., $\omega \to (\omega)^r_n$

Given $f : [\omega]^r \to K$ where $K$ is compact, in what sense can we assert that there is an $H$ homogeneous for $f$?
A generalized notion of a convergent sequence

Definition

Let \( r \in \omega \setminus \{0\} \), \( X \) a space, \( S \subseteq \omega \) infinite and \( f : [S]^r \rightarrow X \), \( f \) converges to \( p \in X \) if for every neighborhood \( U \) of \( p \) there is a finite set \( F \) such that \( f''[S \setminus F]^r \subseteq U \).

1. If \( r = 1 \), then \( f : [S]^1 \rightarrow X \) is a sequence and this notion is the same as usual.

2. If \( (x_n : n \in \omega) \rightarrow p \) and we define \( f \) on \([\omega]^r\) by \( f(s) = x_{\min(s)} \), then \( f \) converges to \( p \).

3. If \( f : [S]^r \rightarrow X \) converges to \( p \) and \( \{s_i : i \in \omega\} \) is pairwise disjoint, then \( (f(s_i) : i \in \omega) \rightarrow p \).
Definition

Given \( r \in \omega \), a space \( X \) is said to be \( r \)-Ramsey if, for each \( f : [\omega]^r \rightarrow X \), there is \( S \subseteq \omega \) infinite such that \( f \upharpoonright [S]^r \) converges. \( X \) has the Ramsey property if it is \( r \)-Ramsey for all \( r \in \omega \).

1. 1-Ramsey \( \iff \) sequentially compact.
2. \( r + 1 \)-Ramsey \( \Rightarrow \) \( r \)-Ramsey
3. Ramsey’s Theorem can be restated as every finite space has the Ramsey property.
Compact metrizable spaces

**Theorem**

*If X is compact metrizable then it has the Ramsey property.*

**Observations:**

1. Applying the theorem to finite $X$, we obtain Ramsey’s classical theorem as a corollary.

$$\forall r, n \in \omega (\omega \rightarrow (\omega)^r_n)$$

2. $r = 1$: Compact metrizable spaces are sequentially compact.

3. $r = 2$: Due to M. Bojańczyk, E. Kopczyński, S. Toruńczyk. Applied to obtain idempotents in compact metrizable semigroups as limits of some particular functions on $[\omega]^2$. 
Proof.

Theorem

If $X$ is compact metrizable then it is $r$-Ramsey for all $r \in \omega$.

Proof: For each $n$ fix a finite cover $\mathcal{U}_n$ by $1/2^n$ balls and let $f : [\omega]^r \to X$. $f$ and $\mathcal{U}_n$ induce a finite coloring of $[\omega]^r$. Using Ramsey’s Theorem, let

$$S_0 \supseteq S_1 \supseteq \ldots S_n \supseteq \ldots$$

so that for all $n$

1. $S_n \subseteq \omega$ is infinite.
2. the diameter of $F_n = f''[S_n]^r$ is less than $1/2^n$

If $p \in \bigcap \{F_n : n \in \omega\}$ and $S \subseteq^* S_n$ for all $n$ then $f \upharpoonright [S]^r$ converges to $p$. \(\dashv\)
Corollaries

If X is compact and the closure of every countable set is first countable, then X has the Ramsey property.

1. Any 1-point compactification of a discrete space is Ramsey
2. and so is any Corson compact,
3. and any compact linearly ordered space.

This can be improved a bit:

Theorem

Sequentially compact spaces of character $< \mathfrak{b}$ have the Ramsey property.
Examples

Let $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ be almost disjoint, $\Psi(\mathcal{A})$ its Isbell-Mrówka space and $K(\mathcal{A})$ its one-point compactification.

Example

If $\mathcal{A}$ is a maximal almost disjoint family, then $K(\mathcal{A})$ is not 2-Ramsey (but is sequentially compact).

Proof: $K(\mathcal{A})$ is $r$-Ramsey if and only if it is $r$-Ramsey with respect to $f : [\omega]^r \to \omega$.

$f : [S]^r \to \omega$ converges to $a \in \mathcal{A}$ if and only if there is $n$ such that

$$f''[S \setminus n]^r \subseteq a$$

$f : [S]^r \to \omega$ converges to $\infty$ if and only if for every $a \in \mathcal{A}$ there is $n$ such that

$$f''[S \setminus n]^r \cap a = \emptyset$$
We may assume \( A \subseteq [\omega \times \omega]^\omega \) and \( \{n\} \times \omega \in A \) for all \( n \).
Define \( f : [\omega]^2 \to K(A) \) by \( f(\{k, n\}) = (k, n) \) \((k < n)\).

Then, for any infinite \( S \subseteq \omega \), and any \( n \)

\[
f''[S \setminus n]^2 \in I^+(A)
\]

**Lemma (Mathias)**
For \( A \) mad, for any decreasing sequence \( B_n \in I^+(A) \) there is \( B \in I^+(A) \) such that \( B \subseteq^* B_n \) for all \( n \).

So, for any \( S \), there is \( A \in A \) such that for all \( n \)

\[
f''[S \setminus n]^2 \cap A \text{ is infinite}
\]

So, no \( f \upharpoonright [S]^2 \) can be convergent. \( \dashv \)
Theorem

The $r$-Ramsey property is preserved under

1. Closed subspaces
2. Continuous images
3. Countable products and $\Sigma$-products

Theorem (van Douwen)

The minimal cardinal $\kappa$ such that $2^\kappa$ is not sequentially compact is the splitting number $s$

$s = \min\{|F| : F \subseteq 2^\omega \text{ is splitting. I.e., for no } A \text{ is } f \upharpoonright A \text{ constant mod finite for all } f \in F \}.$
$2^\kappa$ may be sequentially compact and not Ramsey

**Definition (Blass)**

1. $A$ is **almost homogeneous** for a family of functions $\mathcal{F} \subseteq 2^{[\omega]^r}$ if for each $f \in \mathcal{F}$ there is $n$ such that $f$ is constant on $[A \setminus n]^r$.
2. $\text{par}_r$ is the minimal cardinality of a family of functions $[\omega]^r \to 2$ with no almost homogeneous set.

**Theorem (Blass)**

For each $r \geq 2$, $\text{par}_r = \text{par}_2 = \min\{b, s\}$

Analogous to van Douwen’s characterization of $s$, we have

**Theorem**

$\text{par}_2$ is the minimal cardinal $\kappa$ such that $2^\kappa$ is not $r$-Ramsey.

And so,

$b < s$ implies that $2^b$ is sequentially compact not 2-Ramsey
Theorem

Assuming CH ($\mathfrak{b} = \mathfrak{c}$ should suffice). For each $r$ there is an almost disjoint family $\mathcal{A}$ on $\omega$ such that $K(\mathcal{A})$ is $r$-Ramsey and not $(r + 1)$-Ramsey.

Proof. Build $\mathcal{A} = \{a_\alpha : \alpha \in \omega_1\}$ on $\omega^{r+1}$ starting with

$$\{a_n : n \in \omega\} = \{\{s\} \times \omega : s \in \omega^r\}$$

Not $(r + 1)$-Ramsey will be witnessed by $G$ defined by

$$G(\{k_0, k_1, ..., k_r\}) = (k_0, ..., k_r)$$

$(B_\alpha)_{\alpha}$ enumerate $[\omega]^\aleph_0$ and $(f_\alpha)_{\alpha}$ enumerate all $f : [\omega]^r \rightarrow \omega^{r+1}$

To make the construction work, we need to fix $S_\alpha$ convergent for $f_\alpha$ and add a new $a_\alpha$ witnessing $G \upharpoonright [B_\alpha]^{r+1}$ is not convergent.
Definition

FIN is the ideal of finite subsets of $\omega$.

$FIN^n$ is the Fubini product of $FIN$: defined recursively by

$$X \in FIN^{n+1} \text{ if}$$

$$\{ s \in \omega^n : \{ k : s \circlearrowleft k \in X \} \notin FIN \} \in FIN^n$$

1. $\{ a_n : n \in \omega \} = \{ \{ s \} \times \omega : s \in \omega^r \} \subseteq FIN^{r+1}$,
2. and $a \in FIN^{r+1}$ whenever $a$ is a.d. from all $a_n$
3. $G''[B]^{r+1} \notin FIN^{r+1}$ for any $B$

Lemma

For every $f : [\omega]^r \to \omega^{r+1}$, there is $S \subseteq \omega$ such that

$$f''[S]^r \in FIN^{r+1}$$
(P. Simon): The productivity number for sequential compactness is $\mathfrak{h}$

$\mathfrak{h}$ is the minimal number of mad families needed to split every infinite subset of $\omega$.

If \{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\} witness, then

$$\prod_{\alpha < \mathfrak{h}} K(\mathcal{A}_\alpha)$$

is not sequentially compact.

The productivity number for the Ramsey property is $\geq \mathfrak{h}$

Question

Are there $\mathfrak{h}$ many 2-Ramsey spaces whose product is not 2-Ramsey?